Surface Integrals:

Big Picture:

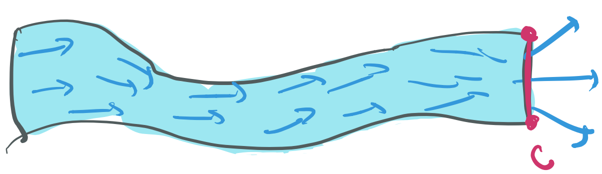
Parameterize surfaces are an extension of parameterized curves.

Surface integrals are an extension of line integrals.

Divergence / Flux form of Green's Theorem

$$\boxed{\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \nabla\cdot\vec{F}\,dA}$$

Intuition for Flux: Think fluid flow



Think water flowing through a river or pipe. The flow of the water is measured by a vector at each point. Say we want to find the amount of water flowing throug c.

This would amount to summing the value in the normal direction along the curve c. I.E. Flow/Flux $= \int_C F\cdot\vec{n}\,ds$

Remember in 2D If $r'(t) = \langle x'(t), y'(t) \rangle$ Then

$$\vec{n}(t) = \frac{\langle y'(t), -x'(t) \rangle}{|r'(t)|}.$$ So,

$$\int_C \vec{F} \cdot \vec{n}\, ds = \int F(r(t)) \cdot \langle y'(t), -x'(t) \rangle\, dt$$

Back to Green's Theorem Let $F = \langle f, g \rangle$

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \int f(t) \cdot y'(t) - g(t) x(t)\, dt$$

$$= \oint_C \langle -g, f \rangle \cdot d\vec{r} \underset{\text{Green's Theorem}}{=} \iint_R f_x + g_y\, dA$$

$$= \iint_R \nabla \cdot \vec{F}\, dA$$

<u>Integration by Parts in higher dimensions</u>

In 1D. $fg\Big|_a^b = \int_a^b \frac{d}{dx}(fg)\, dx = \int_a^b \frac{d}{dx}(f) g + f \frac{d}{dx}(g)\, dx$

Usually written as $\int u\, dv = uv\Big|_a^b - \int_a^b v\, du$

In 2D let $F = \langle fg, 0 \rangle$ Then

$$\int_{\partial R} \vec{F} \cdot \vec{n}\, ds = \iint_R \nabla \cdot F\, dA = \iint_R \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x}\, dA$$

$$\Rightarrow \boxed{\iint_R \frac{\partial f}{\partial x} g\, dx = \int_{\partial R} \vec{F} \cdot \vec{n}\, ds - \iint_R f \frac{\partial g}{\partial x}\, dA}$$

Now This works extending R to all of $\mathbb{R}^2$ to get

$$\iint_{\mathbb{R}^2} \frac{\partial f}{\partial x} g \, dx = -\iint_{\mathbb{R}^2} f \frac{\partial g}{\partial x} dx$$

This is because $\mathbb{R}^2$ has no boundary

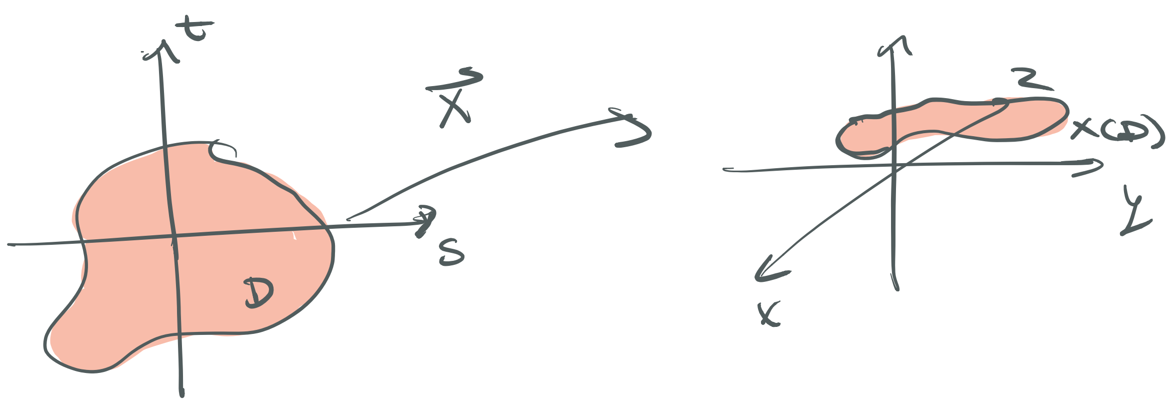
This works for any partial derivatives not just ∂x.

<u>One main use:</u> usually when we have two derivatives

$$\iint_{\mathbb{R}^2} \frac{\partial^2 f}{\partial x^2} f \, dA = \iint_{\mathbb{R}^2} -\left(\frac{\partial f}{\partial x}\right)^2 dA \le 0.$$

<u>Parameterized Surfaces</u>

Let D be a region in $\mathbb{R}^2$ that consists of a connected open set. A **parameterized surface** in $\mathbb{R}^3$ is a continuous function $\vec{X}: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ that is one-to-one on D except possibly along ∂D. We refer to $\vec{X}(D)$ as the underlying surface of X.



Example: Plane formed by the points $(0,0,0)$, $(0,1,1)$ and $(1,0,-1)$.

Exercise: Find a parameterization of a sphere of radius 2, (think spherical coordinates) centered at the origin.

Example:
Parameterizing a raph $z = f(x,y)$.

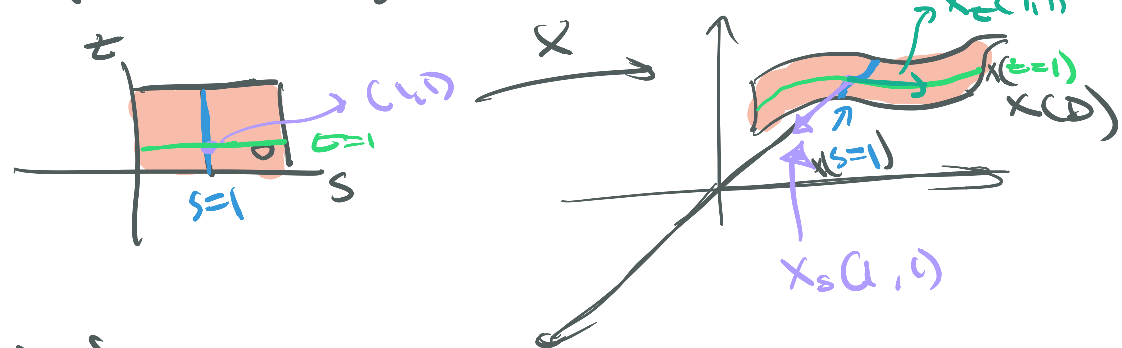
How do we differentiate?
Use partial derivatives and compute component wise.

Let $\vec{x}(s,t) = \langle f(s,t), g(s,t), h(s,t) \rangle$

Then $\vec{x}_s = \langle \frac{\partial f}{\partial s}, \frac{\partial g}{\partial s}, \frac{\partial h}{\partial s} \rangle$

$\vec{x}_t = \langle \frac{\partial f}{\partial t}, \frac{\partial g}{\partial t}, \frac{\partial h}{\partial t} \rangle$

$\vec{x}_s$ gives the tangent vector in the s, direction



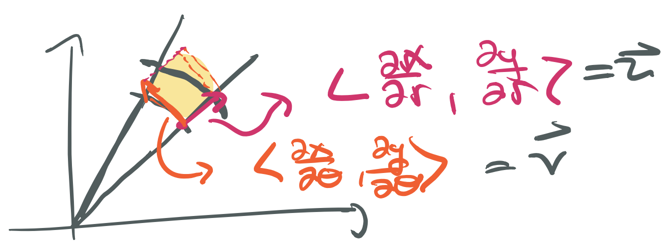
$\vec{x}_s, \vec{x}_t$ form the tangent plane at $x(s,t)$

and $\vec{x}_s \times \vec{x}_t$ gives the normal vector $\vec{n}$.

How do we calculate Surface Area?

Like we did with the change of variables formula

In 2D, for example Polar Coordinates we divide the area into small pieces and estimate the area



Then we thought of $\langle \frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, 0 \rangle$, $\langle \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, 0 \rangle$ as 3D vectors and used Area $= \|\vec{u} \times \vec{v}\|$.

We can do the same thing with surfaces

$$\boxed{\text{Area} = \iint_D \|\vec{x}_s \times \vec{x}_t\| \, dA}$$

These are 3D tangent vectors so $\times$ makes sense.

Example: Calculate the surface area of a sphere of radius a.

Exercise: Find the surface Area of the torus given by

$X(s,t) = \langle (a+b\cos t)\cos s, (a+b\cos t)\sin s, b\sin t \rangle$

Remark:

Given a graph $z = f(x,y)$ we get that the surface area is given by

$$\iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$$

This follows from parameterizing the graph and using $SA = \iint_D \|\vec{x}_s \times \vec{x}_t\|\, ds\, dt$.

We can extend this to integrals of functions over surfaces.

Definition: Let $\vec{X}: D \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth parameterized surface. Denote $S = \vec{X}(D)$. Let $f: S \subset \mathbb{R}^3 \to \mathbb{R}$ be continuous. Then the surface integral of f along $\vec{X}$ is given by

$$\boxed{\iint_{\vec{X}} f \, dS = \iint_D f(\vec{X}(s,t)) \, \|\vec{X}_s \times \vec{X}_t\| \, ds\,dt.}$$

Example:

Evaluate $\iint_{\vec{X}} z^3 \, dS$ where $\vec{X}$ is the sphere of radius a.

Recall: $\|\vec{X}_s \times \vec{X}_t\| = a^2 \sin t$

Definition: Vector Surface Integral

Let F be a vector field, $\vec{X}$ a parameterized surface. Then,

$$\boxed{\iint_{\vec{X}} \vec{F}\cdot d\vec{S} = \iint_{D} \vec{F}(\vec{X}(s,t))\cdot \vec{n}(s,t)\, ds\, dt.}$$

This is the flux across a surface.

Extend the water pipe idea from line integrals
river

Theorem: If $X: D_1 \to \mathbb{R}^3$ and $Y: D_2 \to \mathbb{R}^3$ are both parameterizations of the same surface then

$$\iint_{\vec{X}} f\, dS = \iint_{\vec{Y}} f\, dS$$