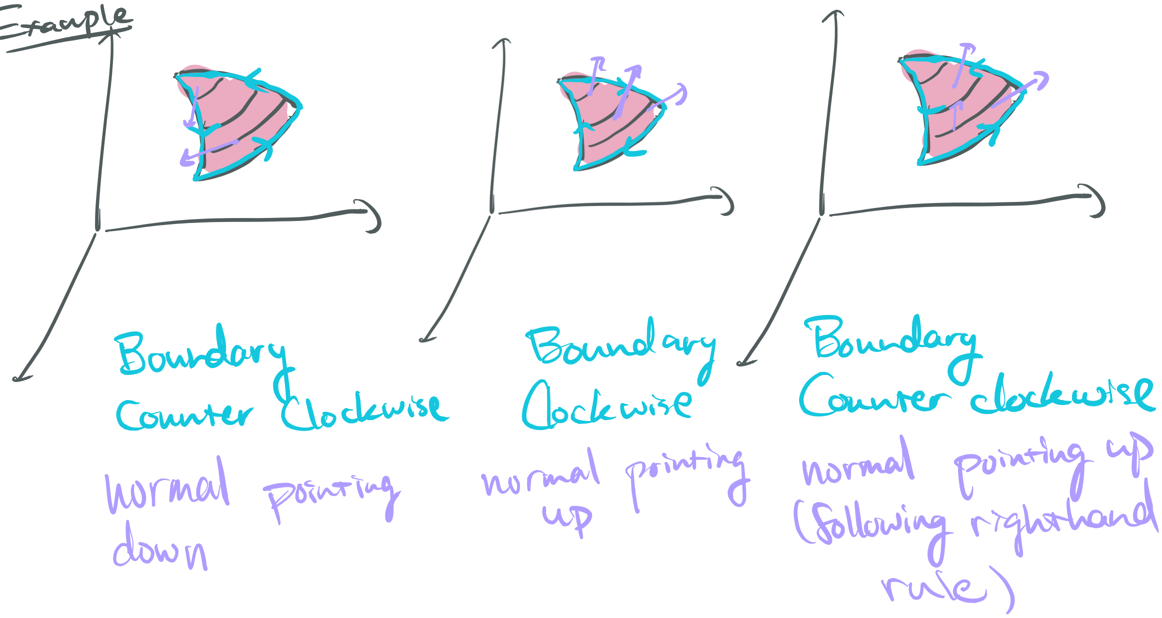
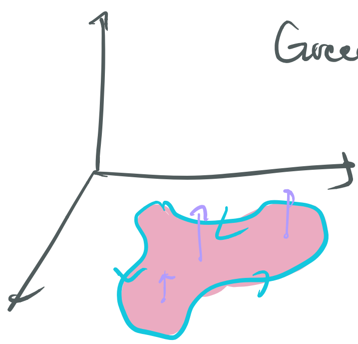
Stoke's Theorem:

With surfaces we have different possible orientations

Example



Green's Theorem:

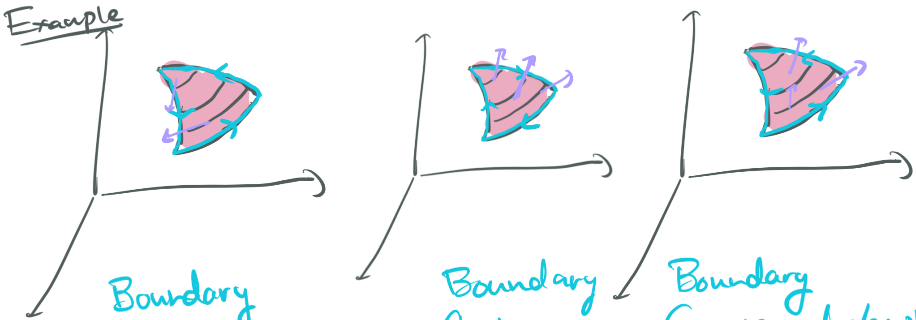


$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times F) \cdot \underline{\vec{k}}\, dA$$

Notice that we can think of the regions from Green's theorem as living in the xy-plane in $\mathbb{R}^3$

Under this interpretation our curve goes counter-clockwise and the normal is pointing up. i.e. it follows the right hand rule with the rotation of the boundary. We say a surface is oriented consistently if the holds

Example



Boundary Counter Clockwise
normal pointing down
not consistent

Boundary Clockwise
normal pointing up
not consistent

Boundary Counter clockwise
normal pointing up (following righthand rule)
consistent

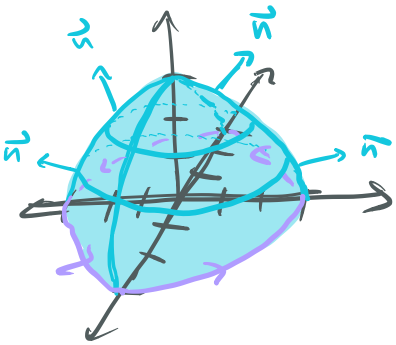
Theorem (Stoke's Theorem)

Let S be a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$. Suppose that ∂S consists of infinitely many piecewise C^1, simple, closed curves each of which is oriented consistently with S. Let F be a vector field of class C^1, whose domain includes S. Then,

$$\boxed{\iint_S (\nabla \times F) \cdot dS = \oint_{\partial S} F \cdot d\vec{r}}$$

Remark: If S is in the xy-plane this is just Green's Theorem. So, this is just an extension of Green's Theorem to more General surfaces.

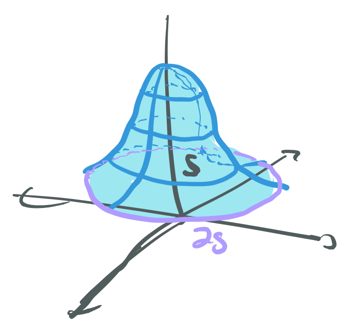
Example: let S be the paraboloid $z=9-x^2-y^2$ defined over the disk in the xy-plane of radius 3. Verify Stoke's Theorem in this case, for $F=\langle 2z-y,\ x+z,\ 3x-2y\rangle$



Example: Let S be defined by the equation $z = e^{-(x^2+y^2)}$ for $z \geq 1/e$.

Let $F = \langle e^{y+z} - 2y, xe^{y+z} + y, e^{x+y} \rangle$.

Calculate $\iint_S (\nabla \times F) \cdot dS$.



Corollary: Suppose S_1, S_2 are two surfaces oriented consistently and such that $\partial S_1 = \partial S_2$ and the orientation of ∂S_1 and ∂S_2 are the same. Then,

$$\boxed{\iint_{S_1} \nabla \times F \cdot dS = \oint_{\partial S_1} F \cdot d\vec{r} = \oint_{\partial S_2} F \cdot d\vec{r} = \iint_{S_2} \nabla \times F \cdot dS}$$

Exercise: Find the work done by the vector field $\vec{F} = \langle xyz - e^x, -xyz, x^2yz + \sin z \rangle$ on a particle that moves along the line segments from $(0,0,0)$ then $(1,1,1)$ then to $(0,0,2)$ then back to $(0,0,0)$