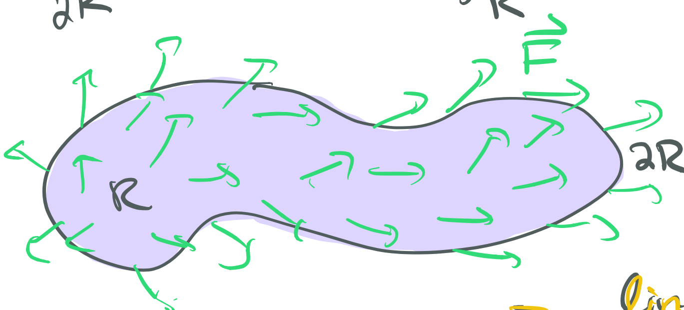
Divergence / Gauss's Theorem

Big Idea: Gauss' theorem gives a way to relate the integral of the divergence inside a solid region with the flux on the surface of the solid region

Recall: The divergence form of Green's theorem was

$$\oint_{\partial R} \vec{F} \cdot \vec{n}\, ds = \iint_R \vec{\nabla} \cdot \vec{F}\, dA$$



line integral

This said that the flux through the boundary of a region in $\mathbb{R}^2$ was equal to the integral dou of the divergence inside of the region. → double integral

Does something similar happen in 3D? surface integral

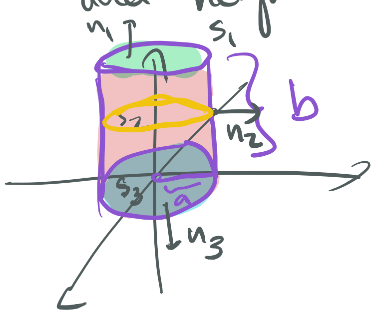
Is the flux through the surface of a 3D object equal the integral of the divergence inside the object? → triple integral

Theorem: (Gauss' Theorem)

Let D be a bounded solid region in $\mathbb{R}^3$ whose boundary ∂D consists of finitely many piecewise smooth, closed oriented surfaces, each oriented so that the unit normals point away from D. Let F be a vector field of class C^1 whose domain includes D. Then

$$\boxed{\oiint_{\partial D} \vec{F} \cdot dS = \iiint_D \nabla \cdot F \, dV}$$

Example: Verify Gauss' Theorem for $F = x\vec{i} + y\vec{j} + z\vec{k}$ and the region D be the solid cylinder of radius a and height b.



a) Compute $\oiint_{\partial D} F \cdot dS$

S_3.

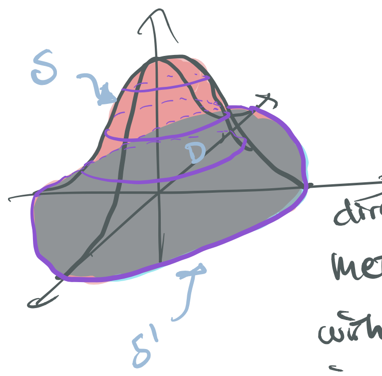
b) Compute $\iiint_D \nabla \cdot F \, dV$

Example: Let $F = e^y\cos(z)\vec{i} + \sqrt{x^3+1}\,\sin z\,\vec{j} + (x^2+y^2+3)\vec{k}$

and let S be the graph

$$z = (1-x^2-y^2)e^{1-x^2-3y^2} \text{ for } z \geq 0$$

oriented by the upward pointing unit normal vector



Compute $\iint F \cdot ds$:

$\iint F\cdot ds$ would be hard to compute directly so we have to come up with a different method. Let's consider the closed surface with the bottom given by the disk $x^2+y^2 \leq 1$ in the xy-plane. Then we can use

Gauss's Theorem.

$$\iint_{\partial D} F\cdot ds = \iint_S F\cdot ds + \iint_{S'} F\cdot ds = \iiint_D \nabla\cdot F\, dV$$

$\nabla\cdot F = 0$ in this case so,

$$\iint_S F\cdot ds = -\iint_{S'} F\cdot ds$$

So we can compute $\iint_S F\cdot ds$ by computing $-\iint_{S'} F\cdot ds$

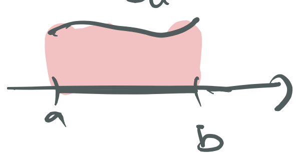
Integrals:

$\int_a^b f(x)\,dx$

single variable

area under a curve

F.T.C $\int_a^b f'(x)\,dx = f(b) - f(a)$



Extension by parameterization

$\longrightarrow$ 2-D integral

Scalar

Line Integral

$\int_C f\,dr$

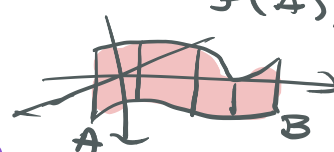
single variable

Vector-form

$\int_C F \cdot d\vec{r}$

F.T.C for Line integrals

$\int_a^b \nabla f \cdot d\vec{r} = f(B) - f(A)$



Flux line integrals

$\int \vec{F} \cdot \vec{n}\,dt$

Extension to 2 variables (xy-plane)

$\iint_R f(x,y)\,dA$

double variable

Volume under a surface

Fubini's Theorem

$\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dy\,dx$

Polar Coordinates

$\iint_R f(r,\theta)\, r\,dr\,d\theta$

These are connected by Green's Theorem

$F = \langle f, g \rangle$

Curl-Form

$\iint_R (g_x - f_y)\,dA = \oint_{\partial R} F\,d\vec{r}$

$= \oint_{\partial R} f\,dx + g\,dy$

Divergence Form

$\iint_R \nabla \cdot F\,dA = \oint_{\partial R} \vec{F} \cdot \vec{n}\,dr$

Extension $\longrightarrow$ to more general

Extension to 2 variables by

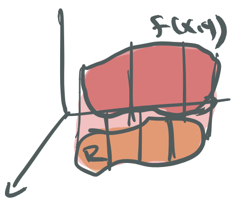
Related by Stoke's Theorem

$\oint_{\partial S} \vec{F}\,d\vec{r} = \iint_S (\nabla \times F) \cdot \vec{n}\,dS$

Change variables

$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(u,v)\left(\frac{\partial x,y}{\partial u,v}\right) du\,dv$$

calculate by iterated integrals



Extension to 3 variable
xyz - axis.

surfaces

Relate to Change of variables

Scalar Surface integral

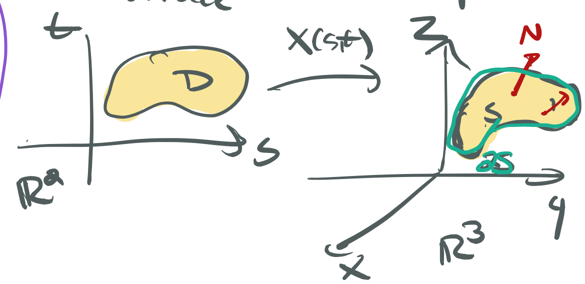
$$\iint_S f\,dS$$

$$= \iint_D f(x(s,t))\,\|\vec{x}_s \times \vec{x}_t\|\,ds\,dt$$

vector surface integrals

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\vec{n}\,ds\,dt$$

vector surface integrals give the flux through the surface



Triple Integral

$$\iiint_R f\,dV$$

This gives 4-D volume "mass" of an object

Cylindrical Coordinates

$$\iiint f(r,\theta,z)\,\underline{r}\,dr\,d\theta\,dz$$

$0 \le r \le \infty,\ 0 \le \theta \le 2\pi,\ -\infty \le z \le -\infty$

Spherical coordinates

$$\iiint f(\rho,\varphi,\theta)\,\underline{\rho^2 \sin\varphi}\,d\rho\,d\varphi\,d\theta$$

$0 \le \rho \le \infty,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi$

General Change of variables

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R'} f(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Fabini's Theorem holds

Calculate using iterated integrals

Connected by Gauss' Theorem

$$\iiint_R \nabla\cdot\vec{F}\,dV = \oiint_{\partial R} \vec{F}\cdot d\vec{S}$$

R is the 3D, ∂R is the surface

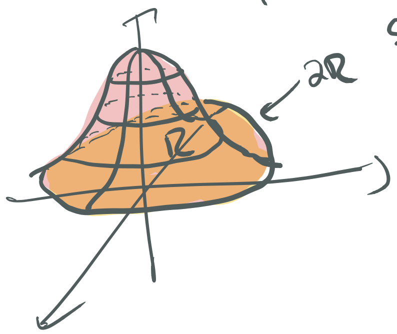
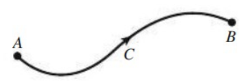
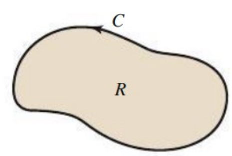
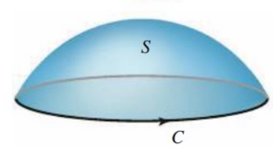


Table 17.4

Fundamental Theorem of Calculus	$\int_a^b f'(x)\,dx = f(b) - f(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$	
Green's Theorem (Circulation form)	$\iint_R (g_x - f_y)\,dA = \oint_C f\,dx + g\,dy$	
Stokes' Theorem	$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$	
Divergence Theorem	$\iiint_D \nabla \cdot \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS$