Lines and planes

June 1, 2020

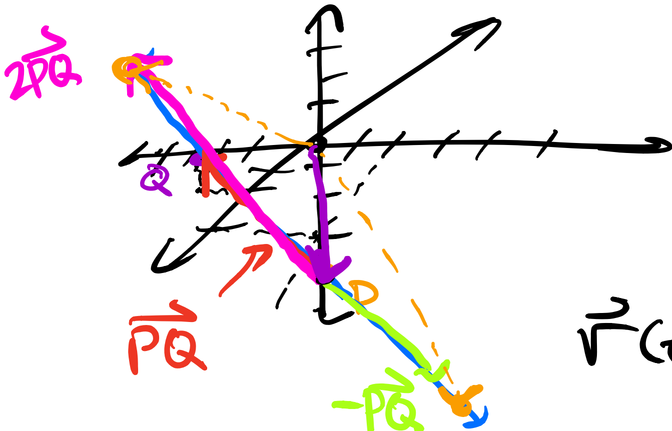
Big Picture for today

Big Picture: 1) Lines can be written in vector form. Two lines may intersect at different “times”.
2) For a plane, need 3 points or a “normal vector” and a point.

Examples

Let $P(3, 2, -1)$ and $Q(2, -2, 1)$. Find a parametric equation for a line through P and Q.

$$t\,\underline{\vec{PQ}} + \underline{P} = \vec{r}(t)$$



$$\vec{PQ} = \langle 2,-2,1\rangle - \langle 3,2,-1\rangle$$
$$= \langle -1,-4,2\rangle$$

$$\vec{r}(t) = t\langle -1,-4,2\rangle + \langle 3,2,-1\rangle$$

Examples

Let $P(3,2,-1)$ and $Q(2,-2,1)$. Find a parametric equation for a line through P and Q.

$$\mathbf{r}(t) = \underset{\vec{x}_0}{\langle 3,2,-1\rangle} + t\,\langle \underset{\vec{a}}{-1},-4,2\rangle\,,$$

or $x = 3 - t$, $y = 2 - 4t$, $z = -1 + 2t$.

Question: Other ways to write?

direction $\langle a,b,c\rangle$ and point (x_0,y_0,z_0)

symmetric form is

$$\vec{r}(t) = \langle 3,2,-1\rangle + 2t\langle -1,-4,2\rangle$$

$$= \langle 3,2,-1\rangle + t\langle -2,-8,4\rangle$$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$t = 3-x$, $t = \frac{1}{2} - \frac{y}{4}$, $t = \frac{1}{2} + \frac{z}{2}$

$$3-x = \frac{1}{2} - \frac{y}{4} = \frac{1}{2} + \frac{z}{2} \rightarrow \text{symmetric form}$$

$$\frac{x-3}{-1} = 3-x$$

Examples

Let $P(3, 2, -1)$ and $Q(2, -2, 1)$. Find a parametric equation for a line through P and Q.

$$\mathbf{r}(t) = \langle 3, 2, -1 \rangle + t \langle -1, -4, 2 \rangle ,$$

or $x = 3 - t$, $y = 2 - 4t$, $z = -1 + 2t$.
Question: Other ways to write?

$$\mathbf{r}_2(t) = \underbrace{\langle 2, -2, 1 \rangle}_{Q} + t \underbrace{\langle +1, +4, -2 \rangle}_{\overrightarrow{QP} = -\overrightarrow{PQ}}$$

starts at Q and goes *backwards* to P ($t = -1$).

Examples

Let $P(3, 2, -1)$ and $Q(2, -2, 1)$. Find a parametric equation for a line through P and Q.

$$\mathbf{r}(t) = \langle 3, 2, -1 \rangle + t \langle -1, -4, 2 \rangle ,$$

or $x = 3 - t$, $y = 2 - 4t$, $z = -1 + 2t$.
Question: Other ways to write?

$$\mathbf{r}_2(t) = \langle 2, -2, 1 \rangle + t \langle -1, -4, 2 \rangle$$

starts at Q and goes *backwards* to P $(t = -1)$.

$$\mathbf{r}_3(t) = \langle 6, 14, -7 \rangle + t \langle -1, -4, 2 \rangle$$

parametrizes same line.

Examples

Let $P(3, 2, -1)$ and $Q(2, -2, 1)$. Find a parametric equation for a line through P and Q.

at

$$\mathbf{r}(t) = \langle 3, 2, -1 \rangle + t \langle -1, -4, 2 \rangle ,$$ P when $t=0$

or $x = 3 - t$, $y = 2 - 4t$, $z = -1 + 2t$.
Question: Other ways to write?

$$\mathbf{r}_2(t) = \langle 2, -2, 1 \rangle + t \langle -1, -4, 2 \rangle$$

starts at Q and goes *backwards* to P ($t = -1$).

$$\mathbf{r}_3(t) = \langle 6, 14, -7 \rangle + t \langle -1, -4, 2 \rangle$$ at P when $t=3$

parametrizes same line. (goes through P at $t = 3$).

Intersections of lines

Different parametrizations $\rightsquigarrow$ 2 lines could go through same point at different “times”.

Intersections of lines

Different parametrizations $\rightsquigarrow$ 2 lines could go through same point at different “times”. **Example:** Let ℓ_1 be the line parametrized by

$$\underline{\mathbf{r}_1(t)} = \langle 1, 2, 3 \rangle + t \langle 2, 3, 4 \rangle$$

and ℓ_2 the line given by

$$\mathbf{r}_2(t) = \langle 7, 3, 3 \rangle + t \langle 2, -1, -2 \rangle$$

Do these lines intersect?

Intersections of lines

Different parametrizations $\rightsquigarrow$ 2 lines could go through same point at different "times". **Example:** Let ℓ_1 be the line parametrized by

$$\mathbf{r}_1(t) = \langle 1, 2, 3 \rangle + t \langle 2, 3, 4 \rangle$$

and ℓ_2 the line given by

$$\mathbf{r}_2(t) = \langle 7, 3, 3 \rangle + t \langle 2, -1, -2 \rangle$$

Do these lines intersect? What about ℓ_1 and ℓ_3:

$$\mathbf{r}_3(t) = \langle 0, 8, -1 \rangle + t \langle 1, 4, 1 \rangle ?$$

Intersections of lines

Different parametrizations $\rightsquigarrow$ 2 lines could go through same point at different "times". **Example:** Let ℓ_1 be the line parametrized by

$$\mathbf{r}_1(t) = \langle 1, 2, 3\rangle + t\langle 2, 3, 4\rangle$$

and ℓ_2 the line given by

$$\mathbf{r}_2(t) = \langle 7, 3, 3\rangle + t\langle 2, -1, -2\rangle$$

Do these lines intersect? What about ℓ_1 and ℓ_3:

$$\mathbf{r}_3(t) = \langle 0, 8, -1\rangle + t\langle 1, 4, 1\rangle?$$

... ℓ_1 and ℓ_4:

$\langle -4, -6, -8\rangle = -2\langle 2, 3, 4\rangle$

$$\mathbf{r}_4(t) = \langle 1, 5, 2\rangle + t\langle -4, -6, -8\rangle?$$

do not intersect

Solution

Need 2 different parameters! Go through same point if $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for some t and s, (not necessarily $t = s$).

Need 2 different parameters! Go through same point if $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for some t and s, (not necessarily $t = s$). Then compare x components:

$\vec{r}_1(t) = \langle x_1(t), y_1(t), z_1(t) \rangle$ $\quad \vec{r}_2(t) = \langle x_2(t), y_2(t), z_2(t) \rangle$

$$x_1(t) = 1 + 2t, \; x_2(s) = 7 + 2s \rightsquigarrow 1 + 2t = 7 + 2s$$

$\vec{r}_1(t) = \langle 1, 2, 3 \rangle + t\langle 2, 3, 4 \rangle = \langle 1+2t, 2+3t, 3+4t \rangle$

or $s = -3 + t$.

$\vec{r}_2(s) = \langle 7, 3, 3 \rangle + s\langle 2, -1, -2 \rangle$

$= \langle 7+2s, 3-s, 3-2s \rangle$

$t = 1 \quad s = -2$

$\vec{r}_1(1) = \langle 1+2, 2+3, 3+4 \rangle = \langle 3, 5, 7 \rangle$

$\vec{r}_2(-2) = \langle 7 + 2(-2), 3-(-2), 3 - 2(-2) \rangle$

$= \langle 3, 5, 7 \rangle$

$=$

Solution

Need 2 different parameters! Go through same point if $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for some t and s, (not necessarily $t = s$). Then compare x components:

$$x_1(t) = 1 + 2t, \;\; x_2(s) = 7 + 2s \rightsquigarrow 1 + 2t = 7 + 2s$$

or $s = -3 + t$. Compare y:

$$y_1(t) = 2 + 3t, \;\; y_2(s) = 3 - s \rightsquigarrow 2 + 3t = 3 - s = 3 - (-3 + t)$$

or $t = 1$. $\rightsquigarrow$ $s = -2$

Solution

Need 2 different parameters! Go through same point if $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for some t and s, (not necessarily $t = s$). Then compare x components:

$$x_1(t) = 1 + 2t, \;\; x_2(s) = 7 + 2s \rightsquigarrow 1 + 2t = 7 + 2s$$

or $s = -3 + t$. Compare y:

$$y_1(t) = 2 + 3t, \;\; y_2(s) = 3 - s \rightsquigarrow 2 + 3t = 3 - s = 3 - (-3 + t)$$

or $t = 1$. $\rightsquigarrow$ $s = -2$ Check z components:

$$z_1(t) = 3 + 4t, \;\; z_2(s) = 3 - 2s$$

Plugging in:

$$z_1(1) = 7, \;\; z_2(-2) = 7$$

$$\vec{r}_1(t) = \langle 1,2,3\rangle + t\langle 2,3,4\rangle = \langle \overset{x_1(t)}{1+2t}, \overset{y_1(t)}{2+3t}, \overset{z_1(t)}{3+4t} \rangle$$

$$\vec{r}_3(s) = \langle 0,8,-1\rangle + s\langle 1,4,1\rangle = \langle \overset{x_2(s)}{s}, \overset{y_2(s)}{8+4s}, \overset{z_2(s)}{-1+s} \rangle$$

$$x_1(t) = x_2(s) \Rightarrow 1+2t = s$$

$$y_1(t) = y_2(s) \Rightarrow 2+3t = 8+4s = 8+4(1+2t)$$

$$2+3t = 8+4+8t \Rightarrow 2 = 12+5t \Rightarrow -10 = 5t$$

$$t = -2 \qquad s = 1+2(-2) = \underline{-3}$$

$$z_1(-2) = 3+4(-2) = 3-8 = \underline{-5}$$

$$z_2(-3) = -1+(-3) = \underline{-4}$$

These lines do not intersect.
Skew.

$$\begin{cases} x_1(t) - x_2(s) = 0 \\ y_1(t) - y_2(s) = 0 \\ z_1(t) - z_2(s) = 0 \end{cases}$$

Solution

Need 2 different parameters! Go through same point if $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for some t and s, (not necessarily $t = s$).
Then compare x components:

$$x_1(t) = 1 + 2t, \;\; x_2(s) = 7 + 2s \rightsquigarrow 1 + 2t = 7 + 2s$$

or $s = -3 + t$. Compare y:

$$y_1(t) = 2 + 3t, \;\; y_2(s) = 3 - s \rightsquigarrow 2 + 3t = 3 - s = 3 - (-3 + t)$$

or $t = 1$. $\rightsquigarrow$ $s = -2$ Check z components:

$$z_1(t) = 3 + 4t, \;\; z_2(s) = 3 - 2s$$

Plugging in:

$$z_1(1) = 7, \;\; z_2(-2) = 7$$

$\rightsquigarrow$ intersect at $\underline{(3, 5, 7)}$. $r_1(1) = (3,5,7) = r_2(-2)$

Exercise

ℓ_1:

$$\mathbf{r}_1(t) = \langle 1, 2, 3 \rangle + t \langle 2, 3, 4 \rangle$$

ℓ_3:

$$\mathbf{r}_3(t) = \langle 0, 8, -1 \rangle + t \langle 1, 4, 1 \rangle ?$$

Intersect?

Exercise

ℓ_1:

$$\mathbf{r}_1(t) = \langle 1, 2, 3 \rangle + t \langle 2, 3, 4 \rangle$$

ℓ_3:

$$\mathbf{r}_3(t) = \langle 0, 8, -1 \rangle + t \langle 1, 4, 1 \rangle?$$

Intersect?
$x_1(t) = 1 + 2t$, $x_2(s) = s$, so $1 + 2t = s$. $y_1(t) = 2 + 3t$, $y_2(s) = 8 + 4s$, so $2 + 3t = 8 + 4(1 + 2t)$, or $t = -2$ and $s = -3$. Plug into z: $z_1(-2) = -5$, $z_2(-3) = -4$. Lines do not intersect.

Definitions

ℓ_1 and ℓ_2 *intersect at one point*.
ℓ_3 does not intersect ℓ_1 and is *skew* - not parallel and not intersecting. ℓ_4 is *parallel* but not equal to ℓ_1.

Planes

Specify a plane by 1) a point on the plane and 2) a perpendicular direction.

Planes

Specify a plane by 1) a point on the plane and 2) a perpendicular direction.
Given a plane, we want an equation that determines if a point $S(x, y, z)$ is on the plane.

Planes

Specify a plane by 1) a point on the plane and 2) a perpendicular direction.
Given a plane, we want an equation that determines if a point $S(x, y, z)$ is on the plane.
If $P(x_1, y_1, z_1)$ is our point, then $\overrightarrow{PS} = \langle x - x_1, y - y_1, z - z_1 \rangle$. If $\mathbf{n} = \langle a, b, c \rangle$ is perpendicular,

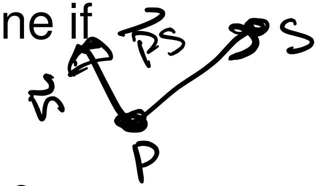
Planes

Specify a plane by 1) a point on the plane and 2) a perpendicular direction.

Given a plane, we want an equation that determines if a point $S(x, y, z)$ is on the plane.

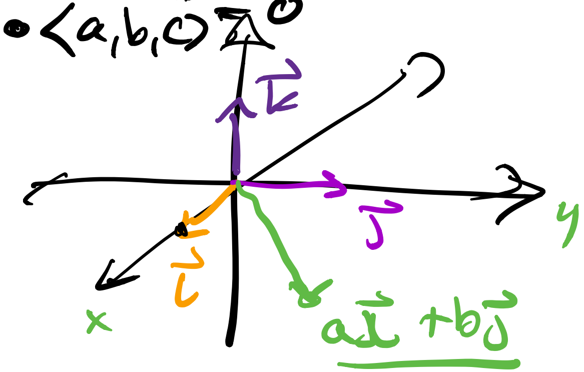
If $P(x_1, y_1, z_1)$ is our point, then $\overrightarrow{PS} = \langle x - x_1, y - y_1, z - z_1 \rangle$. If $\mathbf{n} = \langle a, b, c \rangle$ is perpendicular, then S is in our plane if $\overrightarrow{PS} \cdot \mathbf{n} = 0$, or

P^{initial} $P\ \langle x, y, z \rangle$



$\langle x - x_1, y - y_1, z - z_1 \rangle \cdot \langle a, b, c \rangle = 0$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$



$\vec{i} \cdot \vec{k} = 0$

$\vec{j} \cdot \vec{k} = 0$

$\vec{k} \cdot (a\vec{i} + b\vec{j}) = a(\underbrace{\vec{k} \cdot \vec{i}}_{0}) + b(\underbrace{\vec{k} \cdot \vec{j}}_{0}) = 0.$

Specify a plane by 1) a point on the plane and 2) a perpendicular direction.

Given a plane, we want an equation that determines if a point $S(x, y, z)$ is on the plane.

If $P(x_1, y_1, z_1)$ is our point, then $\overrightarrow{PS} = \langle x - x_1, y - y_1, z - z_1 \rangle$. If $\mathbf{n} = \langle a, b, c \rangle$ is perpendicular, then S is in our plane if $\overrightarrow{PS} \cdot \mathbf{n} = 0$, or

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0. \quad *$$

$\rightsquigarrow$ equation for a plane: $n = \langle a, b, c \rangle$

$$ax + by + cz = d. = ax_1 + by_1 + cz_1 \quad **$$

$ax + by + cz - d$

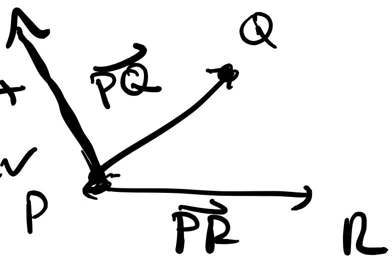
Example

Example

Example: Find the plane determined by the points $P(1, 0, 1)$, $Q(1, 2, 3)$ and $R(2, 3, 2)$.

$\overrightarrow{PQ} \times \overrightarrow{PR}$

perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$



Example

Example: Find the plane determined by the points $P(1,0,1)$, $Q(1,2,3)$ and $R(2,3,2)$.
How to find a normal vector $\mathbf{n}$?

Example

Example: Find the plane determined by the points $P(1,0,1)$, $Q(1,2,3)$ and $R(2,3,2)$.
How to find a normal vector $\mathbf{n}$?
3 distinct points P, Q, R.

Example: Find the plane determined by the points $P(1, 0, 1)$, $Q(1, 2, 3)$ and $R(2, 3, 2)$.
How to find a normal vector $\mathbf{n}$?
3 distinct points P, Q, R. $\rightsquigarrow$ vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in the plane.

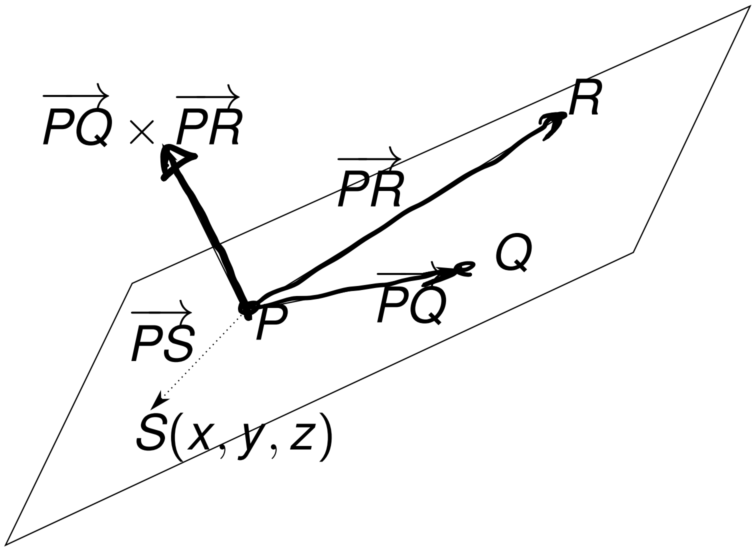


Figure: 3 points, *normal vector* and a point in the plane

$P(1,0,1)$, $Q(1,2,3)$ $R(2,3,2)$

$\vec{QP}$, $\vec{QR}$

$\vec{QP} = \langle 0, -2, -2 \rangle$

$\vec{QR} = \langle 1, 1, -1 \rangle$

$$\vec{QP} \times \vec{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & -2 \\ 1 & 1 & -1 \end{vmatrix} = \vec{i}(4) - \vec{j}(2) + \vec{k}(2)$$

$$= 4\vec{i} - 2\vec{j} + 2\vec{k}$$

$\langle 4, -2, 2 \rangle$

$P=(1,0,1)$, $Q=(1,2,3)$ $R=(2,3,2)$

$\vec{PQ}=(0,2,2)$ $\vec{PR}=(1,3,1)$

$$\vec{PQ}\times\vec{PR}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 2 & 2\\ 1 & 3 & 1\end{vmatrix}=\vec{i}(-4)-\vec{j}(-2)+\vec{k}(-2)$$

$$\vec{PQ}\times\vec{PR}=\langle -4,2,-2\rangle$$

$(\, , \,)$ points, $\langle \, , \, \rangle$ vector

so

$$\langle x-1, y-0, z-1\rangle\cdot\langle -4,2,-2\rangle=0$$

or

$$-4x+2y-2z=-6$$

Using Q

$$\langle x-1, y-2, z-3\rangle\cdot\langle -4,2,-2\rangle=0$$

$$(-4x+4)+(2y-4)+(-2z+6)=0$$

$$-4x+2y-2z=-4+4-6$$

$$-4x+2y-2z=-6$$

Find the plane determined by the points $P(1,2,2)$, $Q(2,3,-1)$, and $R(-2,3,3)$.

$\vec{PQ} = \langle 1, 1, -3 \rangle$

$\vec{PR} = \langle -3, 1, 1 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ -3 & 1 & 1 \end{vmatrix} = \vec{i}(1+3) - \vec{j}(\overset{-8}{1-9}) + \vec{k}(1+3)$$

$$= 4\vec{i} + 8\vec{j} + 4\vec{k}$$

$$\langle x-1, y-2, z-2 \rangle \cdot \langle 4, 8, 4 \rangle = 0$$

Exercise

$$(4x-4)+(8y-16)+(4z-8)=0$$

$$\boxed{4x+8y+4z-28}=0$$

Find the plane determined by the points $P(1,2,2)$, $Q(2,3,-1)$, and $R(-2,3,3)$.

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\langle 1,1,-3\rangle\times\langle -3,1,1\rangle=\langle 4,8,4\rangle,$$

so

$$\langle x-1,y-2,z-2\rangle\cdot\langle 4,8,4\rangle=0,$$

or

$$4(x-1)+8(y-2)+4(z-2)=0$$

$$4x+8y+4z=28$$

$$4x-4+8y-16+4z-8=0$$

$$4x+8y+4z-28=0$$

Intersections of Planes

If $\mathcal{P}$ and $\mathcal{Q}$ are two planes, they are either the same plane, parallel, or intersect at a line.

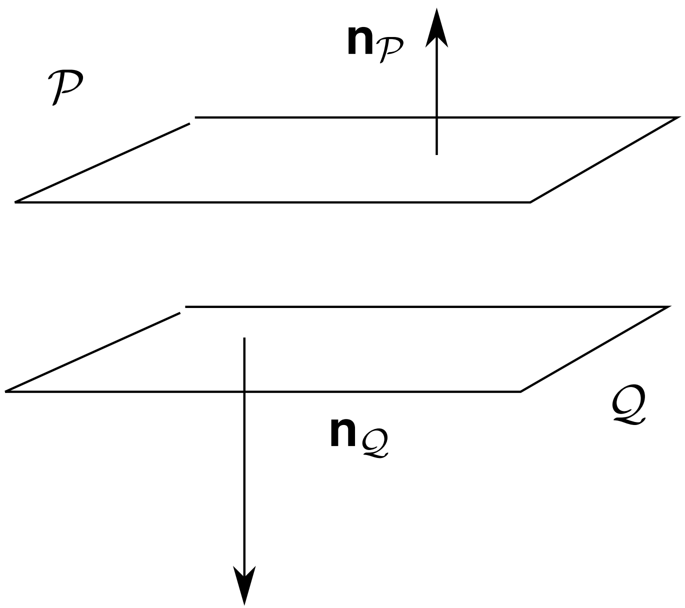


Figure: Two planes with *parallel* normals.

Non-parallel normals $\rightsquigarrow$ planes intersect at a line.

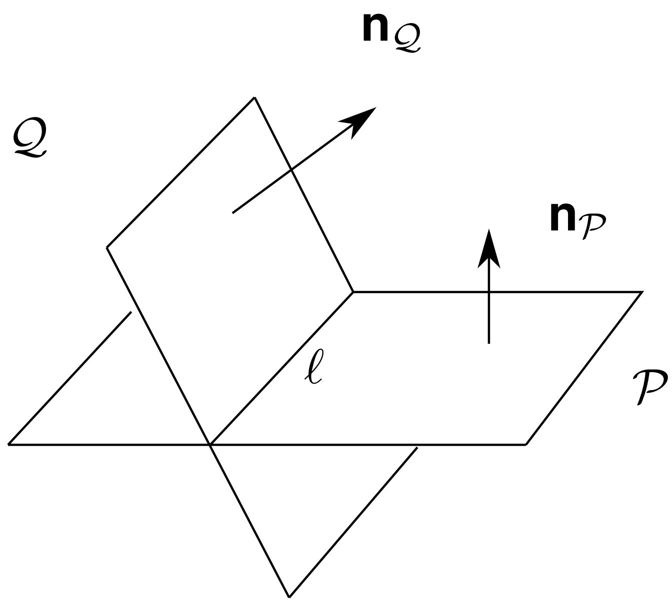


Figure: Two planes with *non-parallel* normals intersect at a line.

Line of intersection

If ℓ is a line in *any* plane $\mathcal{P}$, then its *direction* vector is perpendicular to $\mathbf{n}_{\mathcal{P}}$.

Line of intersection

If ℓ is a line in *any* plane $\mathcal{P}$, then its *direction* vector is perpendicular to $\mathbf{n}_{\mathcal{P}}$. $\rightsquigarrow$ if ℓ lives in *two* planes $\mathcal{P}$ and $\mathcal{Q}$, its direction is perp to *both* $\mathbf{n}_{\mathcal{P}}$ and $\mathbf{n}_{\mathcal{Q}}$.

n_Q

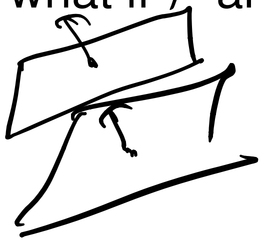
Line of intersection

If ℓ is a line in *any* plane $\mathcal{P}$, then its *direction* vector is perpendicular to $\mathbf{n}_{\mathcal{P}}$. $\rightsquigarrow$ if ℓ lives in *two* planes $\mathcal{P}$ and $\mathcal{Q}$, its direction is perp to *both* $\mathbf{n}_{\mathcal{P}}$ and $\mathbf{n}_{\mathcal{Q}}$.
$\rightsquigarrow$ A direction vector is $\mathbf{n}_{\mathcal{P}} \times \mathbf{n}_{\mathcal{Q}}$.

Line of intersection

If ℓ is a line in *any* plane $\mathcal{P}$, then its *direction* vector is perpendicular to $\mathbf{n}_{\mathcal{P}}$. $\rightsquigarrow$ if ℓ lives in *two* planes $\mathcal{P}$ and $\mathcal{Q}$, its direction is perp to *both* $\mathbf{n}_{\mathcal{P}}$ and $\mathbf{n}_{\mathcal{Q}}$.
$\rightsquigarrow$ A direction vector is $\mathbf{n}_{\mathcal{P}} \times \mathbf{n}_{\mathcal{Q}}$. (Question: what if $\mathcal{P}$ and $\mathcal{Q}$ are parallel?)

$$\vec{n}_P \times \vec{n}_Q = \vec{0}$$



If ℓ is a line in *any* plane $\mathcal{P}$, then its *direction* vector is perpendicular to $\mathbf{n}_{\mathcal{P}}$. $\rightsquigarrow$ if ℓ lives in *two* planes $\mathcal{P}$ and $\mathcal{Q}$, its direction is perp to *both* $\mathbf{n}_{\mathcal{P}}$ and $\mathbf{n}_{\mathcal{Q}}$.
$\rightsquigarrow$ A direction vector is $\mathbf{n}_{\mathcal{P}} \times \mathbf{n}_{\mathcal{Q}}$. (Question: what if $\mathcal{P}$ and $\mathcal{Q}$ are parallel?)
Example: Let $\mathcal{P}$ be the plane given by $2x + 3y - 2z = 4$ and $\mathcal{Q}$ the plane given by $4x - y + 7z = 1$. Find the line of intersection.

$n_P = \langle 2, 3, -2 \rangle$
$n_Q = \langle 4, -1, 7 \rangle$

$$\begin{cases} 2x + 3y = 4 \\ 4x - y = 1 \end{cases}$$

Solution

Solution: Need a direction and a point.

Solution

Solution: Need a direction and a point. for the point fix a z and solve for x and y. For example, $z = 0$ is easy.

Solution

Solution: Need a direction and a point. for the point fix a z and solve for x and y. For example, $z = 0$ is easy. $\rightsquigarrow$

$$\begin{cases} 2x + 3y = 4, \\ 4x - y = 1 \end{cases}$$

Solution: Need a direction and a point. for the point fix a z and solve for x and y. For example, $z = 0$ is easy. $\rightsquigarrow$

$$\begin{cases} 2x + 3y = 4, \\ 4x - y = 1 \end{cases}$$

$$4\left(\tfrac{1}{2}\right) - 1 = y$$

$$y = 2 - 1 = 1$$

$x = 1/2$, $y = 1$.

$$4x - 1 = y$$

$$2x + 3(4x - 1) = 4$$

$$2x + 12x - 3 = 4$$

$$14x = 7$$

$$x = \tfrac{1}{2}$$

Solution

Solution: Need a direction and a point. for the point fix a z and solve for x and y. For example, $z = 0$ is easy. $\rightsquigarrow$

$$\begin{cases} 2x + 3y = 4, \\ 4x - y = 1 \end{cases}$$

$x = 1/2$, $y = 1$. $(1/2, 1, 0)$ is on line.

Solution

Solution: Need a direction and a point. for the point fix a z and solve for x and y. For example, $z = 0$ is easy. $\rightsquigarrow$

$$\begin{cases} 2x + 3y = 4, \\ 4x - y = 1 \end{cases}$$

$x = 1/2$, $y = 1$. $(1/2, 1, 0)$ is on line.
Direction from cross product: $\mathbf{n}_{\mathcal{P}} = \langle 2, 3, -2 \rangle$, $\mathbf{n}_{\mathcal{Q}} = \langle 4, -1, 7 \rangle$

$$\mathbf{n}_{\mathcal{P}} \times \mathbf{n}_{\mathcal{Q}} = \langle 19, -22, -14 \rangle \,,$$

Solution: Need a direction and a point. for the point fix a z and solve for x and y. For example, $z = 0$ is easy. $\rightsquigarrow$

$$\begin{cases} 2x + 3y = 4, \\ 4x - y = 1 \end{cases}$$

$\to y=0$

$x = 1/2$, $y = 1$. $(1/2, 1, 0)$ is on line.
Direction from cross product: $\mathbf{n}_{\mathcal{P}} = \langle 2, 3, -2 \rangle$, $\mathbf{n}_{\mathcal{Q}} = \langle 4, -1, 7 \rangle$

$$\mathbf{n}_{\mathcal{P}} \times \mathbf{n}_{\mathcal{Q}} = \langle 19, -22, -14 \rangle ,$$

Line is given by

$$\mathbf{r}(t) = \langle 1/\underline{2, 1, 0} \rangle + t \, \langle \underline{19, -22, -14} \rangle .$$

Solution

Solution: Need a direction and a point. for the point fix a z and solve for x and y. For example, $z = 0$ is easy. $\rightsquigarrow$

$$\begin{cases} 2x + 3y = 4, \\ 4x - y = 1 \end{cases}$$

$x = 1/2$, $y = 1$. $(1/2, 1, 0)$ is on line.
Direction from cross product: $\mathbf{n}_{\mathcal{P}} = \langle 2, 3, -2 \rangle$, $\mathbf{n}_{\mathcal{Q}} = \langle 4, -1, 7 \rangle$

$$\mathbf{n}_{\mathcal{P}} \times \mathbf{n}_{\mathcal{Q}} = \langle 19, -22, -14 \rangle\,,$$

Line is given by

$2x + 3y - 2z = 4$

$2(39/2) + 3(-21) - 2(-14) = \ldots = 4$

$$\mathbf{r}(t) = \langle 1/2, 1, 0 \rangle + t\,\langle 19, -22, -14 \rangle\,.$$

$4x - y + 7z = 1$ $\quad 4(39/2) - (-21) + 7(-14) = \ldots = 1$

Can check by plugging in!

$r(1) = \langle 19 + 1/2, 1 - 22, 0 - 14 \rangle = \langle 39/2, -21, -14 \rangle$

Exercise

$+0z$

Let $\mathcal{P}$ be the plane given by $x - y = 4$ and $\mathcal{Q}$ the plane given by $2x + y + 2z = -1$. Find line of intersection ℓ.

$\vec{n}_P = \langle 1, -1, 0 \rangle \qquad \vec{n}_Q = \langle 2, 1, 2 \rangle$

Set $y=0$ $\begin{cases} x - 0 = 4 \\ 2x + 0 + 2z = -1 \end{cases} \to \begin{cases} x = 4 \\ 2x + 2z = -1 \end{cases}$

$2z = -1 - 2x$

$z = \frac{-1-2x}{2} = \frac{-1-2(4)}{2} = -9/2$

point is $\langle 4, 0, -9/2 \rangle$

Exercise

Let $\mathcal{P}$ be the plane given by $x - y = 4$ and $\mathcal{Q}$ the plane given by $2x + y + 2z = -1$. Find line of intersection ℓ.

Point $(1, -3, 0)$ is on ℓ. $\mathbf{n}_{\mathcal{P}} = \langle 1, -1, 0 \rangle$ and $\mathbf{n}_{\mathcal{Q}} = \langle 2, 1, 2 \rangle$ so direction is

$$\mathbf{n}_{\mathcal{P}} \times \mathbf{n}_{\mathcal{Q}} = \boxed{\langle -2, -2, 3 \rangle}$$

$$\vec{n}_P \times \vec{n}_Q = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 2 & 1 & 2 \end{vmatrix} = \hat{i}(-2+0) - \hat{j}(2+0) + \hat{k}(1-(-2)) = -2\hat{i} - 2\hat{j} + 3\hat{k}$$

so ℓ is given by

$$\mathbf{r}(t) = \langle 1, -3, 0 \rangle + t \langle -2, -2, 3 \rangle .$$

$$\vec{r}(t) = \langle 4, 0, 9/2 \rangle + t \langle -2, -2, 3 \rangle$$

$$\vec{r}(-3/2) = \langle 1, -3, 0 \rangle + -\tfrac{3}{2} \langle -2, -2, 3 \rangle$$

$$= \langle 1+3,\ -3+3,\ 0-9/2 \rangle$$

$$= \langle \underline{4,\ 0,\ -9/2} \rangle$$