Calculus of vector valued functions

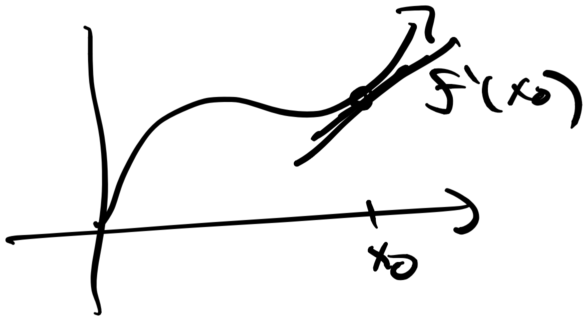
June 1, 2020

Big Picture for today

Big Picture: Vector valued functions look like parametrized curves. We can do calculus with the *components* of the function.

$$y = f(x)$$

Recall: Let f be a function $f : (a, b) \to \mathbb{R}$. If f is differentiable at x_0, then slope is $f'(x_0)$.



Slope

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$\rightsquigarrow$ *tangent line*. If $f(x_0) = y_0$,

$(x_0, f(x_0))$

$$y - y_0 = \underbrace{f'(x_0)}_{\text{slope}}(x - x_0).$$

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Question: What about a curve in space? Start again with line: if $y = mx + b$, then $\vec{\mathbf{r}}(t) = \langle 0, b \rangle + t \langle 1, m \rangle$ gives the same line.

$x = t$
$y = mt + b$

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$$\vec{r}(t) = \langle \overbrace{t}^{f(t)}, \overbrace{b + tm}^{g(t)} \rangle$$

$$f'(t) = 1 \qquad g'(t) = m$$

$$\langle f'(t), g'(t) \rangle = \langle 1, m \rangle$$

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Question: What about a curve in space? Start again with line: if $y = mx + b$, then $\mathbf{r}(t) = \langle 0, b \rangle + t \langle 1, m \rangle$ gives the same line.
Direction vector $\langle 1, m \rangle$ is parallel to line.
Observe: If we differentiate component-wise $\rightsquigarrow$ $\langle 1, m \rangle$ direction vector!

Tangent lines

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$,

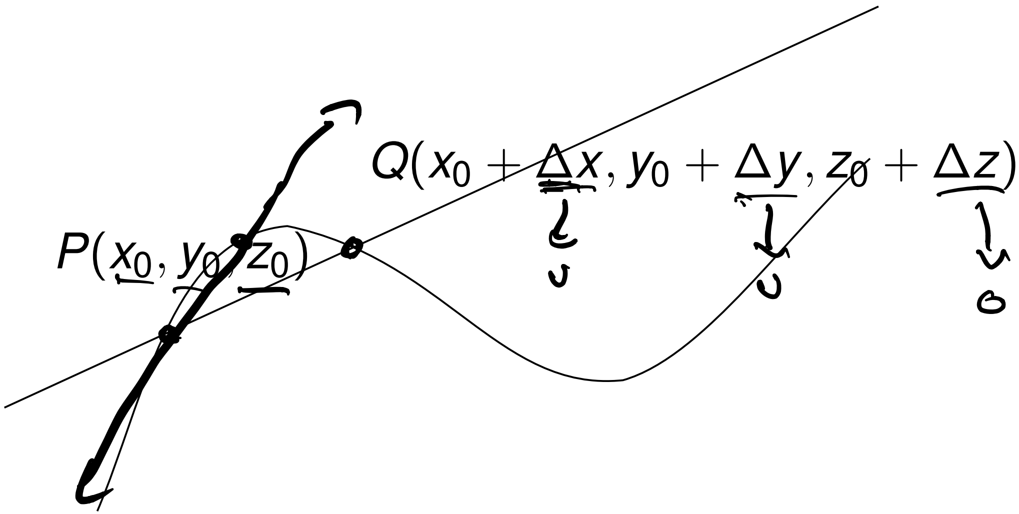

Figure: A curve and two nearby points.

The derivative

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$f(t_0)$

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$$\overrightarrow{PQ} = \langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t) \rangle - \langle f(t), g(t), h(t) \rangle \,,$$

2

Δx, etc. are in terms of t: $x_0 + \Delta x = f(t_0 + \Delta t)$.

$$\overrightarrow{PQ} = \langle f(t + \Delta t), g(t + \Delta t), h(t + \Delta t)\rangle - \langle f(t), g(t), h(t)\rangle\,,$$

$$\rightsquigarrow \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

limit (if it exists) gives *direction* of line through P and Q.

$$\lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} = r'(t)$$

$$= \left\langle \lim_{\Delta t \to 0}\left(\frac{f(t+\Delta t) - f(t)}{\Delta t}\right), \lim_{\Delta t \to 0}\left(\frac{g(t+\Delta t) - g(t)}{\Delta t}\right), \ldots \right.$$

Tangent line example

$= \langle f'(t), g'(t), h'(t) \rangle$

Definition: If f, g, h are differentiable at t, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

gives tangent direction if $\mathbf{r}'(t) \neq \mathbf{0}$.

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Example: In $\mathbb{R}^2$, consider $f(x) = x^2$.

$$f'(x) = 2x$$

$$\text{tangent} = \quad y - \underbrace{y_0}_{= f(x_0)} = f'(x_0)(x - x_0)$$

$$= 2x_0 (x - x_0)$$

$$y = x_0^2 + 2x_0 (x - x_0)$$

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At point $(3, 9)$?

$$y - 9 = 6(x - 3)$$

$$y = 6x - 18 + 9$$
$$y = 6x - 9$$

Example continued

$f(x) = x^2$

$x = t$

$y = x^2 = t^2$

$\vec{r}(t) = \langle t, t^2 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t \rangle$

$\vec{r}\,'(3) = \langle 1, 6 \rangle$

$\vec{r}(3) = \langle 3, (3)^2 \rangle = \langle 3, 9 \rangle$

$\vec{r} = \langle 1, 6 \rangle t + \langle 3, 9 \rangle$

$\vec{r} = \langle t+3, 6t+9 \rangle$

$x = (t+3)$

$y = 6(t+3) - 9$

$= 6t + 18 - 9$

$= 6t + 9$

Parametric: $\mathbf{r}(t) = \langle t, t^2 \rangle$ same curve (why?) Tangent direction $\mathbf{r}'(t) = \langle 1, 2t \rangle$.

Example continued

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$$\mathbf{s}(t) = \mathbf{r}(t_0) + t \langle 1, 2t_0 \rangle$$

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$$\mathbf{s}(t) = \mathbf{r}(t_0) + t \langle 1, 2t_0 \rangle$$

At $(3, 9)$, $t_0 = 3$, write $\mathbf{r}(3) = \langle 3, 9 \rangle$, so

$$\mathbf{s}(t) = \langle 3, 9 \rangle + t \langle 1, 2(3) \rangle$$

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Solve for y in terms of x?

$$\rightsquigarrow y - 9 = 6(x - 3).$$

Let $f(x) = x^3$. 1) Find the tangent line at $(-2, -8)$. Using calc 1
2) write as a vector valued function.
3) Compute the tangent line using this vector valued function.
4) Solve to make sure they are the same line.

$$f'(x) = 3x^2$$

$$y - y_0 = f'(x_0)(x - x_0)$$

$$y = 3(-2)^2(x - (-2)) + (-8)$$

$$y = 12(x+2) - 8$$

Exercise

Let $f(x) = x^3$. 1) Find the tangent line at $(-2, -8)$.
2) write as a vector valued function.
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1) $y = -8 + 3(-2)^2(x+2) = -8 + 12(x+2)$

$x = t$ $\quad\quad$ $\vec{r}(t) = \langle t, t^3 \rangle$

$y = x^3 = t^3$

Let $f(x) = x^3$. 1) Find the tangent line at $(\underline{-2}, \underline{-8})$.
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1) $y = -8 + 3(-2)^2(x+2) = -8 + 12(x+2)$
2) $\mathbf{r}(t) = \langle t, t^3 \rangle$; $\underline{t_0} = -2$

$t = x = -2,$ $\quad r'(t) = \langle 1, 3t^2 \rangle$

$t^3 = y = -8$ $\quad r'(t_0) = r'(-2) = \langle 1, 3(-2)^2 \rangle$

$= \langle 1, 12 \rangle$

$\vec{\ell}(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0)$

$= \langle -2, -8 \rangle + t\langle 1, 12 \rangle$

$= \langle \underbrace{-2+t}_{x}, \underbrace{-8+12t}_{y} \rangle$

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$\vec{s}(t) = \vec{r}(t_0) + t\vec{r}'(t_0)$

$x = -2 + t \qquad y = -8 + 12t$

$y = -8 + 12(x+2)$

$y = -8 + 12((-2+t)+2)$
$= -8 + 12(t)$

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$$\rightsquigarrow y = -8 + 12(x+2)$$

Example: Let $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$. Find the tangent line to this curve at the point $(-1, 0, 3\pi)$.

1) Find t_0 $\quad t = 3\pi = t_0 \quad \cos \vec{t} = -1$

2) Find $\vec{r}\,'(t_0)$.

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\vec{r}(3\pi) = \langle -\sin(3\pi), \cos(3\pi), 1 \rangle$

$= \langle 0, -1, 1 \rangle$

3) $\vec{s}(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0) = \langle -1, 0, 3\pi \rangle + t\langle 0, -1, 1 \rangle$

$= \langle -1, -t, 3\pi + t \rangle$

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Goes through that point at $t = 3\pi$. $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$, so $\mathbf{r}'(3\pi) = \langle 0, -1, 1 \rangle$. Tangent line is

$$\underline{\mathbf{s}(t)} = \langle -1, 0, 3\pi \rangle + t \langle 0, -1, 1 \rangle .$$

$$= \langle -1, 0, 3\pi \rangle - t \langle 0, -1, 1 \rangle$$

1) Find t_0

2) Find $\vec{r}\,'(t_0)$

3) Write $\vec{s}(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0)$

Let $\mathbf{r}(t) = \langle t^2 - 1, t^4 + 2, 3t + 5 \rangle$. Find tangent line through point $P(0, 3, 2)$.

1) $\begin{cases} t^2 - 1 = 0 \\ t^4 + 2 = 3 \\ 3t + 5 = 2 \end{cases}$ $t^2 = 1, \quad t = \pm 1$

$(1)^4 + 2 = 1 + 2 = 3$ ✓

$3(1) + 5 = 8 \neq 2$ ✗

$(-1)^4 + 2 = 1 + 2 = 3$ ✓ $\qquad t_0 = -1$

$3(-1) + 5 = -3 + 5 = 2$ ✓

2) $\vec{r}\,'(t) = \langle 2t, 4t^3, 3 \rangle$, $\vec{r}\,'(-1) = \langle -2, 4(-1)^3, 3 \rangle$
$= \langle -2, -4, 3 \rangle$

3)
$\vec{s}(t) = \langle 0, 3, 2 \rangle + t\langle -2, -4, 3 \rangle = \langle -2t, 3 - 4t, 2 + 3t \rangle$

Exercise

Let $\mathbf{r}(t) = \left\langle t^2 - 1, t^4 + 2, 3t + 5 \right\rangle$. Find tangent line through point $P(0, 3, 2)$. $\mathbf{r}$ goes through P at $t = -1$. $\mathbf{r}' = \left\langle 2t, 4t^3, 3 \right\rangle$, $\rightsquigarrow$ direction $\mathbf{r}'(-1) = \langle -2, -4, 3 \rangle$. $\rightsquigarrow$ line

$$\mathbf{s}(t) = \langle 0, 3, 2 \rangle + t \langle -2, -4, 3 \rangle$$

Exercise

$$\frac{d}{dt}\left(\vec r(2t)\right) = \vec r\,'(2t)\left(\frac{d}{dt}(2t)\right) = 2\vec r\,'(2t) \leftarrow t_0 = -\frac{1}{2}$$

$$|2\vec r\,'(-1)| = |2|\,|r'|$$

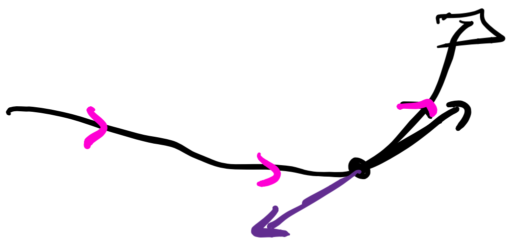
$$\vec r(2t) = \langle (2t)^2 - 1, (2t)^4 + 2, 3(2t)+5 \rangle$$

Let $\mathbf{r}(t) = \langle t^2 - 1, t^4 + 2, 3t + 5\rangle$. Find tangent line through point $P(0, 3, 2)$. $\mathbf{r}$ goes through P at $t = -1$. $\mathbf{r}' = \langle 2t, 4t^3, 3\rangle$, $\rightsquigarrow$ direction $\mathbf{r}'(-1) = \langle -2, -4, 3\rangle$. $\rightsquigarrow$ line

$t = -1$

$$\mathbf{s}(t) = \langle 0, 3, 2\rangle + t\,\langle -2, -4, 3\rangle$$

What about other direction $\langle 2, 4, -3\rangle$?



$$T(t) = \frac{2\vec r\,'(2t)}{|2|\,|r'(2t)|} = \frac{\vec r\,'(2t)}{|r'(2t)|}$$

Unit tangent vector

Often do not need length to specify a line. Let $\mathbf{r}(t)$ be a vector valued function.

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$\vec{r}(t) = \langle t^2 - 1, t^4 + 2, 3t + 5 \rangle$

Note this could be quite messy!

Example: From previous example: $\mathbf{r}' = \langle 2t, 4t^3, 3 \rangle$, so

$$\mathbf{T}(t) = \frac{\langle 2t, 4t^3, 3 \rangle}{(4t^2 + 16t^6 + 9)^{1/2}}$$

so even though $\mathbf{r}'$ is *polynomial*, $\mathbf{T}(t)$ is awful!

Example: Find the unit tangent vector for

$\vec{r}(t) = \langle t^2, 4t, 4\ln(t) \rangle$

$\vec{r}\,'(t) = \langle 2t, 4, \frac{4}{t} \rangle$

$|\vec{r}\,'(t)|^2 = (2t)^2 + (4)^2 + \left(\frac{4}{t}\right)^2$

$= 4t^2 + 16 + \frac{16}{t^2}$

$|r'(t)| = \sqrt{4t^2 + 16 + \frac{16}{t^2}}$

$= \sqrt{4(t^2 + 4 + 4/t^2)}$

$= 2\sqrt{t^2 + 4 + 4/t^2}$

$= 2\sqrt{(t + 2/t)^2} = 2(t + 2/t)$

$(t + 2/t)^2$

$= t^2 + 4t(1/t) + \frac{4}{t^2}$

$= t^2 + 4 + 4/t$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 2t, 4, 4/t \rangle}{2(t + 2/t)}$

$= \left\langle \frac{2t}{2t + 4/t}, \frac{4}{2t + 4/t}, \frac{4/t}{2t + 4/t} \right\rangle$

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$|r'(t)| = 3t^2$

Example: $\mathbf{r}(t) = \langle \cos(t^3), \sin(t^3)\rangle$ has $\mathbf{r}'(t) = \langle -3t^2 \sin(t^3), 3t^2 \cos(t^3)\rangle$ and $\mathbf{T}(t) = \langle -\sin(t^3), \cos(t^3)\rangle$.

$|r'(t)|^2 = (-3t^2 \sin(t^3))^2 + (3t^2 \cos(t^3))^2 = (3t^2)^2(\sin(t^3)^2 + \cos(t^3)^2)$

$(-3t)^2 = (3t)^2$

circle is just $\vec{r}(t) = \langle \cos(t) \ \sin(t) \rangle$

1 rotation is $\underline{2\pi}$ distance

$2\pi \cdot 2 \cdot r$

$t = 0$ to $t = 2\pi$ is 2π unit

$\vec{r}\,'(t) = \langle -\underline{\sin(t)}, \ \underline{\cos(t)} \rangle$

Often do not need length to specify a line. Let $\mathbf{r}(t)$ be a vector valued function.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

$\vec{r}'(t) \neq \vec{0}$

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Example: $\mathbf{r}(t) = \langle \cos(t^3), \sin(t^3) \rangle$ has $\mathbf{r}'(t) = \langle -3t^2 \sin(t^3), 3t^2 \cos(t^3) \rangle$ and $\mathbf{T}(t) = \langle -\sin(t^3), \cos(t^3) \rangle$.
$\mathbf{T}(t)$ is used in some kinds of geometry.

Properties of the derivative

- $\mathbf{c}$ constant vector has $\frac{d}{dt}\vec{\mathbf{c}} = \vec{\mathbf{0}}$
- $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ (sum) $\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$
- $\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ (scalar product rule)
- $\frac{d}{dt}\mathbf{u}(f(t)) = \mathbf{u}'(f(t))f'(t)$ (chain rule)
- $\frac{d}{dt}(\underbrace{\mathbf{u}(t) \cdot \mathbf{v}(t)}_{\text{scalar}}) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$ (vector dot product rule)
- $\frac{d}{dt}(\underbrace{\mathbf{u}(t) \times \mathbf{v}(t)}_{\text{vector}}) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ (cross product rule)

Integration of vector valued functions

We take derivatives componentwise, so we take anti-derivatives componentwise:

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Definition: Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$. If there is a vector function $\mathbf{R}(t) = \langle F(t), G(t), H(t) \rangle$ such that $F' = f$, $G' = g$, and $H' = h$, then $\mathbf{R}$ is an *anti-derivative* of $\mathbf{r}$.

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Example: Does $\mathbf{r}(t) = \left\langle t^3, \sin(t), t\cos(t^2)\right\rangle$ have an anti-derivative?

$$F(t) = \int t^3\,dt = \frac{1}{4}t^4 + C_1$$

$$G(t) = \int \sin(t)\,dt = -\cos(t) + C_2$$

du

$$H(t) = \frac{1}{2}\int 2t\cos(t^2)\,dt = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u) + C_3$$

$$= \frac{1}{2}\sin(t^2) + C_3$$

$u = t^2$

$du = 2t$

Integration of vector valued functions

We take derivatives componentwise, so we take anti-derivatives componentwise:

Definition: Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. If there is a vector function $\mathbf{R}(t) = \langle F(t), G(t), H(t)\rangle$ such that $F' = f$, $G' = g$, and $H' = h$, then $\mathbf{R}$ is an *anti-derivative* of $\mathbf{r}$.

Example: Does $\mathbf{r}(t) = \left\langle t^3, \sin(t), t\cos(t^2)\right\rangle$ have an anti-derivative?

$$\mathbf{R}(t) = \left\langle t^4/4 + c_1, -\cos(t) + c_2, \frac{1}{2}\sin(t^2) + c_3 \right\rangle$$

$$\vec{R}(t) = \left\langle t^4/4, -\cos t, \tfrac{1}{2}\sin(t^3)\right\rangle + \langle \underline{c_1, c_2, c_3} \rangle$$

$$= \left\langle t^4/4, -\cos t, \tfrac{1}{2}\sin(t^3)\right\rangle + \vec{C}$$

We take derivatives componentwise, so we take anti-derivatives componentwise:

Definition: Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$. If there is a vector function $\mathbf{R}(t) = \langle F(t), G(t), H(t) \rangle$ such that $F' = f$, $G' = g$, and $H' = h$, then $\mathbf{R}$ is an *anti-derivative* of $\mathbf{r}$.

Example: Does $\mathbf{r}(t) = \left\langle t^3, \sin(t), t\cos(t^2) \right\rangle$ have an anti-derivative?

$$\mathbf{R}(t) = \left\langle t^4/4 + c_1, -\cos(t) + c_2, \frac{1}{2}\sin(t^2) + c_3 \right\rangle$$

New thing! The *constant* of integration is a constant *vector*. Can label it $\mathbf{c}$.

Find anti-derivatives:

use trig identity

Half-angle Formula

Integrate by part

Factor and then inverse use a trig integral

$$\mathbf{r}(t) = \left\langle \sin^2(t), te^{-t}, \frac{1}{t^2+2t+2} \right\rangle$$

$$\sin(\theta/2) = \sqrt{\frac{1-\cos\theta}{2}} \rightarrow \sin^2(\theta/2) = \frac{1-\cos(\theta)}{2}$$

$$\sin^2(t) = \frac{1}{2} - \frac{\cos(2t)}{2}$$

$$\int \sin^2 t \, dt = \int \frac{1}{2} - \frac{\cos(2t)}{2} \, dt$$

$u = 2t$

$du = 2dt$

$$= \frac{1}{2}t - \frac{1}{2}\frac{1}{2}\int \cos(2t)2dt$$

$$= \frac{1}{2}t - \frac{1}{4}\int \cos(u)du$$

$$= \frac{1}{2}t - \frac{1}{4}\sin(2u) + C_1 = \frac{1}{2}t - \frac{1}{4}\sin(2t) + C_1$$

$$\int t e^{-t}\,dt$$

$$\int u\,dv = uv - \int v\,du$$

$u = t \qquad v = -e^{-t}$

$du = dt \qquad dv = e^{-t}\,dt$

$$\int t e^{-t}\,dt = (-t e^{-t}) - \int -e^{-t}\,dt$$

$$= (-t e^{-t}) - (+e^{-t}) + C_2$$

$$= -t e^{-t} - e^{-t} + C_2$$

$$\int \frac{1}{t^2+2t+2}\,dt = \int \frac{1}{1+(t^2+2t+1)}\,dt$$

$$= \int \frac{1}{1+(t+1)^2}\,dt$$

$u = t+1$

$du = dt$

$$= \int \frac{1}{1+u^2}\,du$$

$$= \arctan(u) + C_3$$

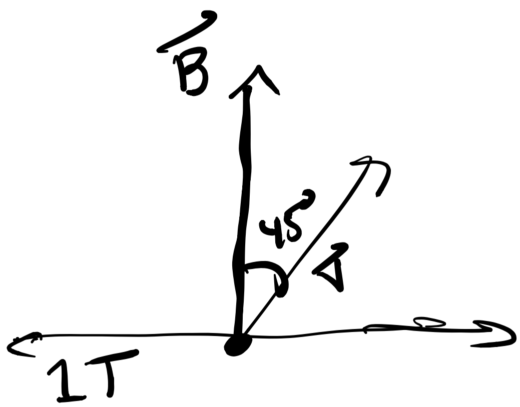
$$= \arctan(t+1) + C_3$$

Exercises

Find anti-derivatives:

$$\mathbf{r}(t) = \left\langle \sin^2(t), te^{-t}, \frac{1}{t^2 + 2t + 2} \right\rangle$$

$$\mathbf{R}(t) = \langle (1/2)t - (1/4)\sin(2t), -te^{-t} - e^{-t}, \arctan(t+1) \rangle + \tilde{\mathbf{c}}$$



$$\vec{F} = |q|\,|\underline{B \times \vec{v}}| = |q|\,|\vec{B}|\,|\vec{v}|\,\underbrace{\sin(\theta)}_{1}$$

$45°$

$$= |-1.6\times10^{-19}|\,|2|\,|4\times10^5|\,\underbrace{\tfrac{\sqrt{2}}{2}}_{\sin(45°)}$$

$$\int \underline{\sin^2(x)}\,dx = \tfrac{1}{3}\sin^3(x)$$

$\int u^2\,du$

$$\frac{d}{dx}\left(\tfrac{1}{3}\sin^3(x)\right) = 3\sin^2(x)\cos(x)$$

$u = \sin(x)$

$du = \cos(x)\,dx$

$$-1.6\times10^{-19} \cdot 2 \cdot \tfrac{1}{2} \cdot 4\times10^5$$

$$= 6.4\times10^{-14}$$