Arc length and Curvature

June 24, 2020

Big Picture for today

Big Picture: We can use calculus to find the length of a curve. We can use arc length to parametrize curves so we can understand the shape.

Arc length of functions

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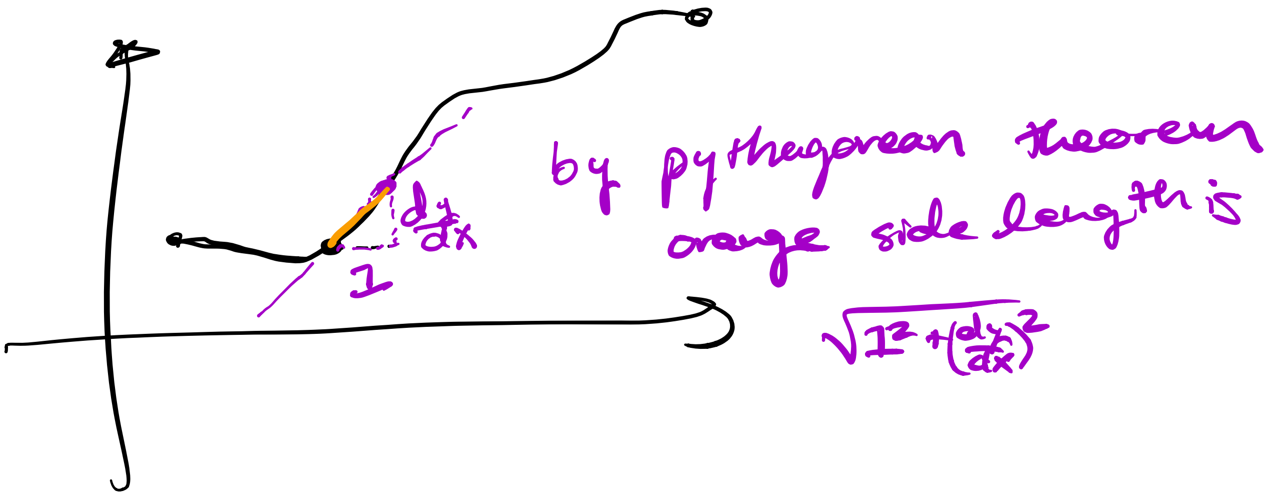

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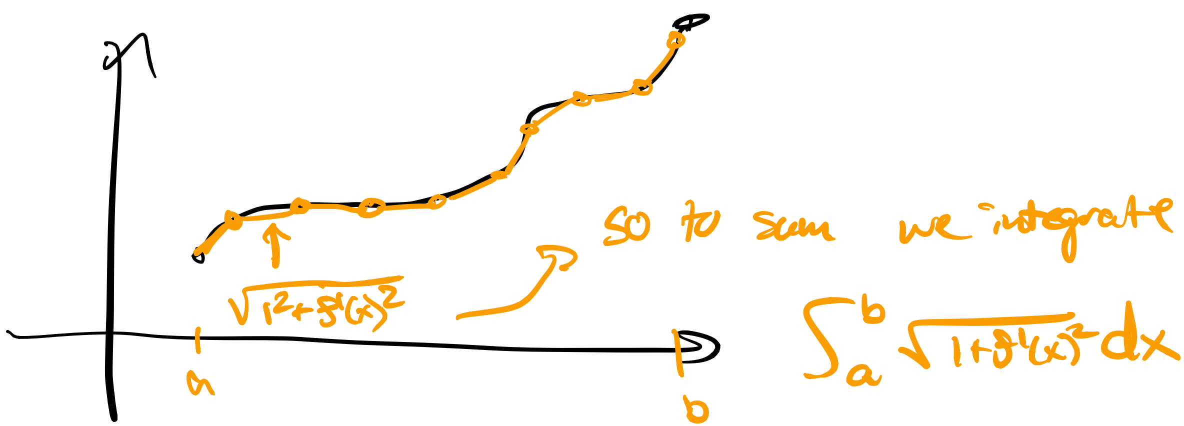

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$$L = \int_0^5 \sqrt{1^2 + f'(x)^2} dx$$

We want to extend this idea to vector valued functions and higher dimensions

Arc length of curves

Recall that we can turn $y = f(x)$ into a vector valued function. The easiest way is to take $x = t, y = f(t)$ so our vector valued function becomes $\mathbf{r}(t) = < t, f(t) >$, Notice that

$$|\mathbf{r}'(t)| = | < 1, f'(t) > | = \sqrt{1 + f'(t)^2}.$$

This agrees with our length we used to find the arc length of a graph.

Arc length of curves

We can use this idea to extend arc length to all vector valued function $\mathbf{r}(t)$. The arc length of a curve from $t = a$ to $t = b$ is given by

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In three dimensions if $\mathbf{r}(t) =< f(t), g(t), h(t) >$ then

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt$$

Let's do an example where we can check that the formula is correct. Recall that $\mathbf{r}(t) = < a\cos(t), a\sin(t) >$ parameterizes a circle of radius a. From geometry we know that the circumference should be $2\pi a$. So the arc length from $t = 0$ to $t = 2\pi$ should be $2\pi a$.

$$L = \int_0^{2\pi} |\vec{r}'| \, dt \qquad r'(t) = \langle -a\sin(t), a\cos(t) \rangle$$

$$|r'(t)|^2 = (-a\sin(t))^2 + (a\cos(t))^2 = a^2\sin^2 t + a^2\cos^2 t$$

$$= a^2(\sin^2 t + \cos^2 t)$$

$$= a^2$$

$$\rightarrow |r'(t)| = a$$

$$L = \int_0^{2\pi} a \, dt = at\Big|_0^{2\pi} = 2\pi a - 0a = 2\pi a \checkmark$$

Exercise

Find the arc length of the curve $\mathbf{r}(t) = < 4\sin(t), t^2, 4\cos(t) >$ from $t = 0$ to $t = 6$.

$$L = \int_0^6 |r'(t)|\,dt$$

$$r'(t) = \langle 4\cos(t),\ 2t,\ -4\sin(t) \rangle$$

$$|r'(t)|^2 = (4\cos t)^2 + (2t)^2 + (-4\sin t)^2$$

$$= 4 + 4t^2 = 4(1+t^2)$$

$$|r'(t)| = 2\sqrt{1+t^2}$$

$$\Rightarrow L = \int_0^6 2\sqrt{1+t^2}\,dt = 2\int_0^6 \sqrt{1+t^2}\,dt$$

$$= 2\left[\frac{1}{2}\left(t\sqrt{1+t^2} + \ln\left|t+\sqrt{1+t^2}\right|\right)\right]_0^6 = 6\sqrt{1+6^2} + \ln\left|6+\sqrt{1+6^2}\right|$$

$$= 6\sqrt{37} + \ln\left|6+\sqrt{37}\right|$$

$$\int \sqrt{1+t^2}\,dt =$$

$$u = \sqrt{1+t^2} \qquad v = t$$
$$du = \frac{t}{\sqrt{1+t^2}} \qquad dv = dt$$

$$= t\sqrt{1+t^2} - \int \frac{t^2}{\sqrt{1+t^2}}\,dt$$

$$= t\sqrt{1+t^2} - \int \frac{1+t^2-1}{\sqrt{1+t^2}}\,dt$$

$$= t\sqrt{1+t^2} - \int \sqrt{1+t^2}\,dt + \int \frac{1}{\sqrt{1+t^2}}\,dt$$

$$\Rightarrow 2\int \sqrt{1+t^2}\,dt = t\sqrt{1+t^2} + \int \frac{1}{\sqrt{1+t^2}}\,dt$$

$$t = \sec(\theta) \Rightarrow dt = \sec\theta \tan\theta \, d\theta$$

$$\Rightarrow \int \frac{1}{\sqrt{1+t^2}}\,dt = \int \frac{1}{\sqrt{1+\sec^2\theta}} \sec\theta \tan\theta \, d\theta$$

$\tan\theta = \sec\theta \tan\theta$

$$= \int \frac{1}{\tan\theta} \sec\theta \tan\theta \, d\theta$$

$$= \int \sec\theta \, d\theta = \int \sec\theta \left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right) d\theta$$

set $u = \sec\theta + \tan\theta \Rightarrow du = \sec^2\theta + \sec\theta\tan\theta \, d\theta$

$$\Rightarrow = \int \frac{1}{u}\,du = \ln|u| + c$$

$$= \ln|\sec\theta + \tan\theta|$$

$$\ln|t + \sqrt{1+t^2}|$$

$\tan^2\theta = 1 + \sec^2\theta$

$\Rightarrow \sqrt{1+t^2} = \tan\theta$

Parameterizing by Arc length

We can define an arc length function for a curve $\mathbf{r}(t)$ by

$$s(t) = \int_a^t |\mathbf{r}'(x)| dx$$

Using the fundamental theorem of calculus we get that

$$\frac{ds}{dt}(t) = \frac{d}{dt} \int_a^t |\mathbf{r}'(x)| dx = |\mathbf{r}'(t)|.$$

Now $|\mathbf{r}'(t)| \neq 0$ so (by the inverse function theorem) we can solve for t in terms of s. Although, this can be impossible to do by hand. But the key point is that given any curve we can re-parameterize it in terms of arc length. I.e. we can rewrite it as $\mathbf{r}(t(s))$ so that $\int_0^a |\mathbf{r}'(t(s))| ds = a$.

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Why do we want to parameterize by Arclength? It gives us an **intrinsic** parameter; which means it depends only on how the curve itself bends, and not how fast the curve is traced. Why?

$$\mathbf{r}'(s(t)) = \mathbf{r}'(s)\frac{ds}{dt} = \mathbf{r}'(s)|\mathbf{r}'(t)|.$$

Hence

$$\mathbf{r}'(s) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

So, $\mathbf{r}(t)$ has unit speed.

Parameterize $\mathbf{r}(t) = < \cos(t^2), \sin(t^2) >$ by arc length.

$r'(t) = \langle -2t\sin(t^2), 2t\cos(t^2) \rangle$

$|r'(t)| = \sqrt{4t^2\sin^2(t^2) + 4t^2\cos^2(t^2)}$

$= \sqrt{4t^2}$

$= 2t$

$s = \int_0^t |r'(a)|\,da = \int_0^t 2a\,da = a^2\Big|_0^t = t^2$

$\Rightarrow s = t^2$

$\Rightarrow \vec{r}(s) = \langle \cos(s), \sin(s) \rangle$ ✓

Exercise

Parameterize $\mathbf{r}(t) =< e^t, e^t, e^t >$ by arc length.

$$\vec{r}'(t) = \langle e^t, e^t, e^t \rangle$$

$$|r'(t)| = \sqrt{e^{2t} + e^{2t} + e^{2t}} = (\sqrt{3})e^t$$

$$s = \int |r'(t)| dt = \int_0^t \sqrt{3} e^a \, da = \sqrt{3} e^a \Big|_0^t = \sqrt{3} e^t - \sqrt{3}$$

$$\Rightarrow \sqrt{3} + \left(\frac{s}{\sqrt{3}}\right) = e^t \Rightarrow t = \ln(s/\sqrt{3} + \sqrt{3})$$

$$\Rightarrow \vec{r}(s) = \langle e^{\ln(s/\sqrt{3}+\sqrt{3})}, e^{\ln(s/\sqrt{3}+\sqrt{3})}, e^{\ln(s/\sqrt{3}+\sqrt{3})} \rangle$$

$$= \langle s/\sqrt{3}, s/\sqrt{3}, s/\sqrt{3} \rangle + \langle \sqrt{3}, \sqrt{3}, \sqrt{3} \rangle$$

Curvature

Recall, in Calculus one the second derivative is associated with acceleration. The same is true for vector valued function.

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Recall, in Calculus one the second derivative is associated with acceleration. The same is true for vector valued function. For a curve $\mathbf{r}(t)$ we have the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$. In 1-dim acceleration was only related to a change of speed but notice since this is a vector, we get acceleration if there is a change in direction.
Now, when studying curves we want to find a way to distinguish curves independent of parameterization. We can do this by looking at how they change direction, i.e. acceleration. The issue is that parameterization determines our acceleration. To avoid this issue we re-parameterize by arc length.

Curvature

Let $\mathbf{r}(s)$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$ is the unit tangent, the curvature is

$$\kappa(s) = \left| \frac{d\mathbf{T}}{ds} \right|$$

Typically we will not have a curve parameterized by arclength. In this case, we can use the formula

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

where $\mathbf{T}$ is the unit tangent vector

Alternative Fomula

If we let $\mathbf{v} = \mathbf{r}'$ and $\mathbf{a} = \mathbf{v}'$ then

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

There is a proof of why this is true in the book. Sometimes this formula will be easier to use

Example

Any definition of curvature should tell us that a straight line does not curve, so let's check that a line has 0 curvature.

line $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ $\quad \vec{v} = \langle a, b, c \rangle$

$\vec{r}\,'(t) = \vec{v}$ $\quad |\vec{v}| = \sqrt{a^2+b^2+c^2}$

$$\vec{T}(t) = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}} \right\rangle$$

$\frac{dT}{dt} = \langle 0, 0, 0 \rangle$ since T is independent of time.

$\Rightarrow \left| \frac{dT}{dt} \right| / |r'(t)| = 0.$

Exercise

Find the curvature of a circle of radius 1. Hint: From the shape of a circle we should expect the curvature to be constant.

Circle: $\vec{r}(t) = \langle \cos t, \sin(t) \rangle$

$\vec{r}'(t) = \langle -\sin(t), \cos(t) \rangle$

$|\vec{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t} = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1$

This implies $t = s$ so it's already parameterized by arclength

$\Rightarrow \vec{T}(t) = \vec{T}(s) = \vec{r}'(t) = \langle -\sin(t), \cos(t) \rangle$

$\kappa = \left| \frac{d\vec{T}}{ds}(s) \right| = \left| \langle -\cos(t), -\sin(t) \rangle \right| = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1}$

$= 1.$

Bonus: Curvature of a circle of radius r.

$= \frac{1}{r}$. In this case $s = \frac{t}{r}$ so we end up with $\frac{1}{r}$ instead of 1.

Unit Normal

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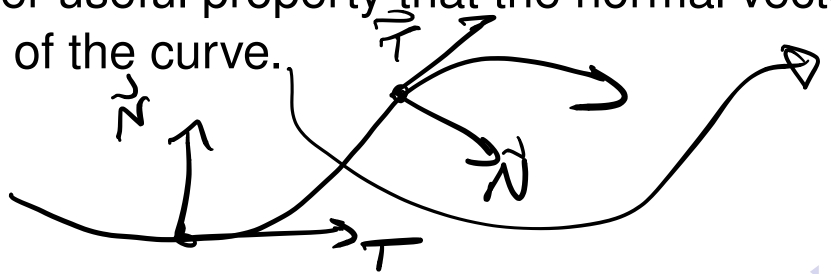
Another useful property that the normal vector points to the inside of the curve.



This is useful for min/max problems later.

Find the unit normal for a circle of radius *r*.

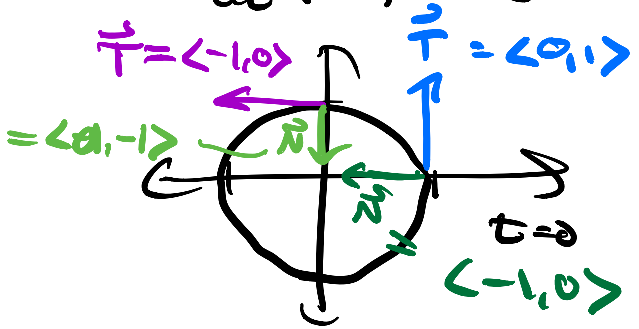
$$\vec{r}(t) = \langle r\cos(t), r\sin(t)\rangle$$

$$\vec{r}\,'(t) = \langle -r\sin(t), r\cos t\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = \sqrt{r^2} = r$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin(t), \cos(t)\rangle$$

$$\vec{N}(t) = \frac{d}{dt}\vec{T}(t) = \langle -\cos(t), -\sin(t)\rangle$$



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For example, we can find a helix and a circle with the same curvature and if we rotate a circle slightly we could get a curve with the same tangent and normal vectors at $t = 0$.

helix = $\langle a\cos t, a\sin t, t\rangle$

$\vec{r}'(t) = \langle -a\sin t, a\cos t, t\rangle$

$|\vec{r}'(t)| = \sqrt{a^2 + 1^2}$

$\Rightarrow \vec{T}(t) = \frac{1}{\sqrt{a^2+1}} \langle -a\sin t, a\cos t, 1\rangle$

$\vec{T}'(t) = \frac{1}{\sqrt{a^2+1}} \langle -a\cos t, -a\sin t, 0\rangle$

$|\vec{T}'(t)| = \frac{a^2}{\sqrt{1+a^2}}$

$\Rightarrow k = \frac{|T'(t)|}{|r'(t)|} = \frac{a}{\sqrt{1+a^2}} \Big/ \sqrt{1+a^2} = \frac{a}{1+a^2} = \frac{1}{2}$

$a = \frac{1}{2} + \frac{1}{2}a^2 \Rightarrow a^2 - 2a + 1 = 0$

$\Rightarrow (a-1)^2 = 0 \Rightarrow \boxed{a=1}$

Note $|r'(t)| = \sqrt{1+a^2}$ = constant $\Rightarrow s = \frac{t}{\sqrt{1+a^2}}$

Also, $|T'(t)| = \frac{a}{\sqrt{1+a^2}} \Rightarrow |\vec{N}(t)| = \langle -\cos t, -\sin t, 0\rangle$

$\vec{B}(t) = \vec{T} \times \vec{N} = \left[\langle -\sin t, \cos t, 1\rangle \times \langle -\cos t, -\sin t, 0\rangle\right] \frac{1}{\sqrt{1+a^2}}$

$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin & \cos & 1 \\ -\cos & -\sin & 0 \end{vmatrix} = (\vec{i}\,\sin + \vec{j}(-\cos) + \vec{k}(1))$

$\Rightarrow \vec{B}(s) = \left\langle \sin\left(t/\sqrt{1+a^2}\right), -\cos\left(s/\sqrt{1+a^2}\right), 0 \right\rangle \cdot \frac{1}{\sqrt{1+a^2}}$

$$\frac{d\vec{B}}{ds}(s) = \frac{1}{1+a^2} \langle \cos(s/\sqrt{1+a^2}),\ \sin(s/\sqrt{1+a^2}),\ 0 \rangle$$

$$\tau = -\frac{d\vec{B}}{ds}(s) \circ \vec{N} = -\frac{1}{1+a^2} \langle \cos(\_\_),\ \sin(\_\_),\ 0 \rangle \circ \langle -\cos(\_\_),\ \sin(\_),\ 0 \rangle$$

$$= \frac{1}{1+a^2}$$

Torsion of a circle is 0.

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For example, we can find a helix and a circle with the same curvature and if we rotate a circle slightly we could get a curve with the same tangent and normal vectors at $t = 0$.
To help differentiate between two curves we need another idea.
The unit binormal vector is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

Torsion

We want to see how **B** changes. We define torsion, τ, by

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

We can derive a formula for **B**:

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$$

Results about curvature and torsion

A line is the only curve with 0 curvature.
Circle and helices are the only curves with constant curvature and torsion.
Any curve with zero torsion lies in a plane.