Limits and continuity

June 28, 2020

Big Picture for today

Big Picture:
For a limit to exist in dimensions 2 and 3, must get the same limit along any direction.

Limits

Recall: For a single variable function $f : (a, b) \to \mathbb{R}$, we have 3 kinds of limits, the limit, the left-hand limit, and the right-hand limit.

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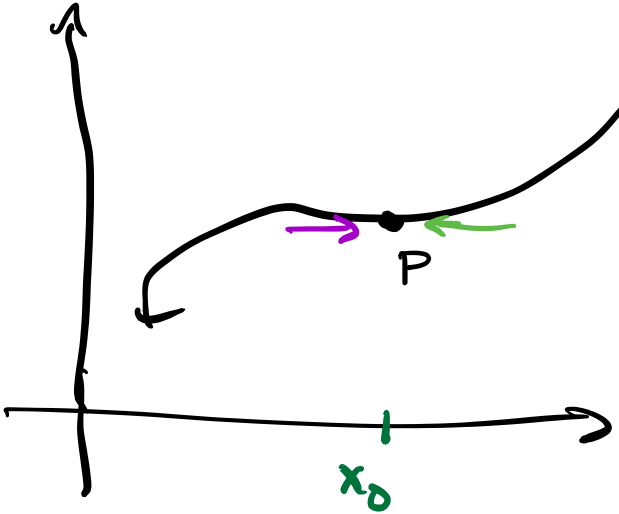

$f(x)$

$\lim_{x \to x_0} f(x)$ exists if

$$\lim_{x \to \underline{x_0^+}} f(x) = \lim_{x \to x_0^-} f(x)$$

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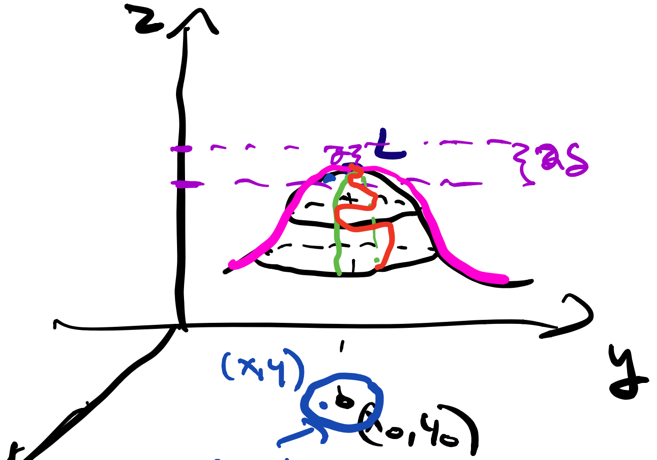
Definition: Let D be a domain in $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a function. For $P_0(x_0, y_0)$ we say the function f has limit L at P_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $P(x, y)$ satisfies $0 < |\langle x - x_0, y - y_0\rangle| < \delta$, then $|f(x, y) - L| < \epsilon$.



Circle of radius δ about (x_0, y_0)

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In this case, we write

$$\lim_{(x,y)\to(x_0,y_0)} f = \lim_{P\to P_0} f(x, y) = L$$

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Definition: Let D be a domain in $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a function. For $P_0(x_0, y_0)$ we say the function f has limit L at P_0 if for every $\underline{\epsilon >}$ 0 there exists $\underline{\delta >}$ 0 such that if $P(x, y)$ satisfies $0 < |\langle \underline{x - x_0}, \underline{y - y_0} \rangle| < \delta$, then $|\underline{f(x, y)} - \underline{L}| < \epsilon$.

In this case, we write

0.0001, $\delta = 0.000000002$

$$\lim_{(x,y)\to(x_0,y_0)} f = \lim_{P\to P_0} f(x, y) = L$$

Intuitively: the limit exists if P close to P_0 implies f is close to L.

Note: The function f need not be defined at (x_0, y_0). If the function is defined at (x_0, y_0) and $f(x_0, y_0) = L$, the f is *continuous* at (x_0, y_0).

Examples

Example: Let $f = \alpha x + \beta y$ be linear. Domain is $\mathbb{R}^2$ and at each point $P_0(x_0, y_0)$ we have

$$\lim_{(x,y)\to(x_0,y_0)} f = \alpha x_0 + \beta y_0.$$

δ depends on ε

$\varepsilon/(|\alpha|+|\beta|)$

Given $\varepsilon > 0$

Assume $0 < |\langle x - x_0, y - y_0\rangle| < \delta$. Then we need to show $|f(x,y) - (\alpha x_0 + \beta y_0)| < \varepsilon$

$|x - x_0| < \delta$, $|y - y_0| < \delta$

$a = 2, -2 \le 2 + |-2| = 4$

$|a+b| \le |a| + |b|$

$4 = |2+2| \le 4$

$$|f(x,y) - (\alpha x_0 + \beta y_0)| = |\alpha x + \beta y - (\alpha x_0 + \beta y_0)|$$

$$= |\alpha x - \alpha x_0 + \beta y - \beta y_0| = |\alpha(x - x_0) + \beta(y - y_0)|$$

$$\le |\alpha(x - x_0)| + |\beta(y - y_0)|$$

$$= |\alpha|\,|x - x_0| + |\beta|\,|y - y_0| < |\alpha|\delta + |\beta|\delta = (|\alpha| + |\beta|)\delta < \varepsilon$$

Examples

$\delta(1+|\alpha|+|\beta|)$ $\frac{\varepsilon}{\cancel{(|\alpha|+|\beta|)}}$

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$$\lim_{(x,y)\to(x_0,y_0)} f = \alpha x_0 + \beta y_0.$$

Proof: Idea: make $\underline{x - x_0}$ and $\underline{y - y_0}$ show up.

$$|\langle \underline{x-x_0}, y-y_0\rangle|^2 \to 0$$

$$|\langle x-x_0, y-y_0\rangle|^2 = (x-x_0)^2 + (\underline{y-y_0})^2$$

$$\geq \underline{(x-x_0)^2}$$

$$|\underline{x-x_0}| \to 0 \qquad |\underline{y-y_0}| \to 0$$

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Proof: Idea: make $x - x_0$ and $y - y_0$ show up. For any (x, y), we have

$|\underline{f(x,y) - L}| \to 0$ as $|\langle x-x_0, y-y_0\rangle| \to 0$

$$|f(\underline{x, y}) - (\alpha \underline{x_0 + \beta y_0)}| = |\alpha \underline{(x - x_0) + \beta(y - y_0)}|.$$

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$$|f(x, y) - (\alpha x_0 + \beta y_0)| = |\alpha(x - x_0) + \beta(y - y_0)|.$$

If (x, y) is close to (x_0, y_0) then $|x - x_0|$ and $|y - y_0|$ are both small.

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$$|f(x,y) - (\alpha x_0 + \beta y_0)| = |\alpha(\underline{x - x_0) + \beta(y -} y_0)|.$$

If (x, y) is close to (x_0, y_0) then $|x \underline{- x_0}|$ and $|y \underline{- y_0}|$ are both small. ⇝ use triangle inequality:

$$|a+b| \le |a| + |b|$$

$$\begin{aligned} |\alpha\underbrace{(x - x_0)}_{a} + \underbrace{\beta(y - y_0)}_{b}| &\le |\alpha(x - x_0)| + |\beta(y - y_0)| \\ &\le |\alpha||\underline{x - x_0}| + |\beta||\underline{y - y_0}| \end{aligned}$$

Continued

We also have $|\langle x - \underline{x_0, y - y_0}\rangle| = ((x - x_0)^2 + (y - y_0)^2)^{1/2}$, so

$|\langle x - x_0, y - y_0\rangle| \geq |x \underline{- x_0}|$ and also $\geq |y \underline{- y_0}|$.

Continued

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Then

$$|\alpha||x-x_0|+|\beta||y-y_0| \leq |\alpha||\langle x-x_0, y-y_0\rangle|+|\beta||\langle x-x_0, y-y_0\rangle|,$$

Continued

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Then

$$|\alpha|\underline{|x-x_0|} + |\beta|\underline{|y-y_0|} \leq |\alpha|\underline{|\langle x-x_0, y-y_0\rangle|} \underset{\checkmark}{+} |\beta|\underline{|\langle x-x_0, y-y_0\rangle|},$$

or

$$|\underbrace{f(x,y) - (\overset{L}{\alpha x_0 + \beta y_0})}_{\to 0}| \leq (|\alpha| + |\beta|)|\langle x \underbrace{- x_0, y - y_0}_{\to 0}\rangle| \to 0$$

as $(x,y) \to (x_0, y_0)$.

Exercises

Example: Let $f(x,y) = x^2 + y^2$. Prove that

$(2)^2+(3)^2 = 13$

$$\lim_{(x,y)\to(2,3)} f = \underset{L}{\underline{13}}.$$

$|x - x_0|$ & $|y-y_0|$

$|\langle x-x_0, y-y_0\rangle| < \delta$

$\Rightarrow |x-2| < \delta$

$|y-3| < \delta$

$|x+2| < 4+\delta$

$|y+3| < 6+\delta$

$$|f(x,y) - L| = |x^2+y^2-13|$$

$$= |x^2-4+y^2-9|$$

$$= |(x+2)\underbrace{(x-2)}_{x-x_0} + \underbrace{(y-3)}_{y-y_0}(y+3)|$$

$$\le |(x+2)(x-2)| + |(y-3)(y+3)|$$

$$\le |x+2||x-2| + |y-3||y+3|$$

$$\le (4+\delta)(\delta) + \delta(6+\delta)$$

$$= 2\delta^2 + 10\delta < \varepsilon$$

$$\le 2\delta + 10\delta = 12\delta < 12\cdot\varepsilon/12 = \varepsilon$$

$|y+3|$

$= |y-3+6|$

$\le |y-3| + 6$

$\le \delta + 6$

$(\frac{1}{2})^2 = \frac{1}{4} \le \frac{1}{2}$

$\delta < 1$ then $\delta^2 < \delta$

Given $\varepsilon > 0$ take

$\delta < \varepsilon/12$

Exercises

Example: Let $f(x, y) = x^2 + y^2$. Prove that

$$\lim_{(x,y)\to(2,3)} f = 13.$$

Exercise: Let $f(x, y) = xy$. Prove that

$$\lim_{(x,y)\to(3,-2)} f = -6.$$

$|\langle x-3, y+2\rangle| < \delta$

$|x-3| < \delta$

$|y+2| < \delta$

y is close to -2

$-2+\delta$

$-2-\delta$

$|f(x,y)+6| < \varepsilon$

$(x-3)(y+2) = xy - 3y + 2x - 6$

$$|f(x,y)+6| = |xy+6|$$

$$= |(x-3)(y+2) + 3\overset{(-2+\delta)\,(-2-\delta)}{y} - 2x + 6 + 6|$$

$$= |(x-3)(y+2) + 3(y+2) - 6 - 2(x-3) - 6 + 6 + 6|$$

$$= |(x-3)(y+2) + 3(y+2) + -2(x-3)|$$

$$\leq |(x-3)(y+2)| + |3(y+2)| + |-2(x-3)|$$

$$\leq |\underset{\delta}{(x-3)}||\underset{\delta}{y+2}| + 3|\underset{\delta}{y+2}| + 2|\underset{\delta}{x-3}|$$

$$\leq \delta^2 + 3\delta + 2\delta$$

$\delta < 1$ then $\delta^2 < \delta$

$(1/2)^2 = \frac{1}{4} < 1/2$

$$= \boxed{\delta^2 + 5\delta}$$

$$\leq \delta + 5\delta = \underline{6\delta} < 6\underline{(\varepsilon/6)} = \underline{\varepsilon}$$

Given $\varepsilon > 0$, choose $\underline{\delta < 1}$ s.t. $\underline{\delta < \varepsilon/6}$

$|f(x,y) + 6| \leq 6\underline{\delta < \varepsilon}$

$\Rightarrow |f(x,y) + 6| < \varepsilon.$

$\delta < \varepsilon/6$

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$$
\begin{aligned}
|\underbrace{xy-(-6)}_{\to 0}| &= |(x-3+3)(y+2-2)+6| \\
&= |(x-3)(y+2-2)+3(y+2-2)+6| \\
&= |(x-3)(y+2)-2(x-3)+3(y+2)-6+6| \\
&\le \underbrace{|x-3|}_{\to 0}\underbrace{|y+2|}_{\to 0} + 2\underbrace{|x-3|}_{\to 0} + 3\underbrace{|y+2|}_{\to 0} \\
&\to 0
\end{aligned}
$$

as $\underline{|\langle x-3, y+2\rangle| \to 0}.]$

Another method

Idea: Roughly, the limit exists if for every continuous parametrized curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ with $\mathbf{r}(\underline{t_0}) = \langle \underline{x_0, y_0}\rangle$ we have

$$\lim_{t\to t_0} \underline{f(x(t), y(t))} = L$$

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If the (single variable) limit along *every possible curve* exists and equals L, then the limit exists.

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Problem: How do you check *every possible curve*?

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If the (single variable) limit along *every possible curve* exists and equals L, then the limit exists.

Problem: How do you check *every possible curve*? You don't.

This is used to prove a limit *does not exist*: If there are (at least) 2 paths along which there are different limits, then the limit does not exist.

An example

Example: For $(x, y) \neq (0, 0)$, let

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

$$f(0,0) = \frac{0 \cdot 0}{0^2 + 0^2} = \frac{0}{0}$$

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Proof: Find (at least) two paths: start with linear $\underline{\mathbf{r}(t) = \langle at, bt \rangle}$.

$\vec{r}(0) = \langle 0,0 \rangle$ $\lim_{t\to 0} f(\vec{r}(t))$

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Then

$f(\vec{r}(t))$

$$f(x(t), y(t)) = \frac{abt^2}{a^2t^2 + b^2t^2} = \frac{ab}{a^2 + b^2}.$$

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For different values of a and b we get different limits. (For example, $a = 1$, $b = 1$ vs. $a = 1$, $b = -1$.)

$\vec{r}(t) = \langle t, t\rangle$

$a=1 \quad b=1 \qquad \lim_{t\to 0} \underline{f(\vec{r}(t))} = \frac{1\cdot 1}{1^2+1^2} = \underline{1/2} \neq$

$a=1 \quad b=-1 \qquad \lim_{t\to 0} \underline{f(\vec{r}(t))} = \frac{1\cdot(-1)}{(1)^2+(-1)^2} = \underline{-1/2}$

$\vec{r}(t) = \langle t, -t\rangle$

Exercises

$$\lim_{t\to0}\frac{\sin(t)}{t} = \frac{\sin(0)}{0} = \frac{0}{0}$$

$$\lim_{t\to0}\frac{\sin t}{t} = \lim_{t\to0}\frac{\frac{d}{dt}(\sin t)}{\frac{d}{dt}(t)} = \lim_{t\to0}\frac{\cos t}{1} = \frac{1}{1} = 1.$$

Exercise: For $(x, y) \neq (0, 0)$, let

$$f(x, y) = \frac{(x-2y)^2}{x^2+y^2}.$$

$\vec{r}(t) = \langle 0, t-1\rangle$ $\quad \lim_{t\to1} f(\vec{r}(t)) = f(\langle 0,0\rangle)$

Prove that $\lim_{(x,y)\to(0,0)} f$ does not exist.

Hint: try $\vec{r}(t) = \langle at, bt\rangle$ $\quad a=0 \quad b=1.$

$\vec{r}(t) = \langle 0, t\rangle$ $\quad \langle t,t\rangle, \langle -t,t\rangle, \langle t,-t\rangle$

$$\lim_{t\to0} f(0, t) = \lim_{t\to0}\frac{(-2t)^2}{0^2+t^2} = \lim_{t\to0}\frac{4t^2}{t^2} = 4.$$

\#

$\vec{r}(t) = \langle t, 0\rangle$

$$\lim_{t\to0} f(t, 0) = \lim_{t\to0}\frac{t^2}{t^2+0^2} = \lim_{t\to0}\frac{t^2}{t^2} = 1$$

Exercises

Exercise: For $(x, y) \neq (0, 0)$, let

$$f(x, y) = \frac{(x - 2y)^2}{x^2 + y^2}.$$

Prove that $\lim_{(x,y)\to(x_0,y_0)} f$ does not exist.
Example: For $(x, y) \neq (0, 0)$, let

$$f(x, y) = \frac{xy^2}{\underline{x^2} + y^4}.$$

Prove $\lim_{(x,y)\to\underline{(0,0)}} f$ does not exist.

$$\vec{r}(t) = \langle \underline{t^2, t} \rangle$$

$$\lim_{t\to 0} \vec{r}(t) = \langle 0, 0 \rangle$$

$$\lim_{t\to 0} f(\vec{r}(t)) = \lim_{t\to 0} f(t^2, t)$$

$$= \lim_{t\to 0} \frac{(t^2)(t)^2}{(t^2)^2 + (t)^4}$$

$$= \lim_{t\to 0} \frac{t^4}{2t^4} = \underline{\frac{1}{2}}$$

$\vec{r}(t) = \langle t, t \rangle$ $\quad \lim_{t\to 0} \vec{r}(t) = \langle 0, 0 \rangle$

$\vec{r}(t) = \underline{\langle -t, t \rangle}$

$$\lim_{t\to 0} f(\vec{r}(t)) = \lim_{t\to 0} f(\underset{(-t)}{t}, t) = \lim_{t\to 0} \frac{-t(t)^2}{t^2 + t^4}$$

$(-t, t)$

$$= \lim_{t\to 0} \frac{-t^3}{t^2 + t^4}$$

$$= \lim_{t\to 0} \frac{-t^3}{t^2(1+t^2)}$$

$$= \lim_{t\to 0} \frac{-t}{1+t^2} = 0$$