

Partial derivatives

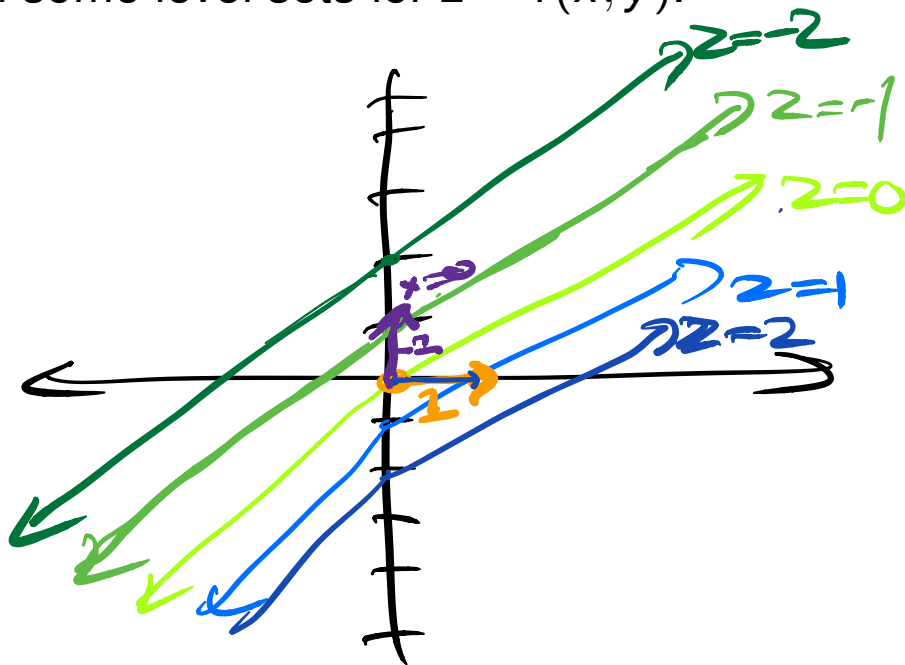
June 24, 2020

Big Picture: Partial derivatives tell you how a function changes in one direction.

Partial derivatives: motivation

Let $f(x, y) = x - y$. Sketch some level sets for $z = f(x, y)$.

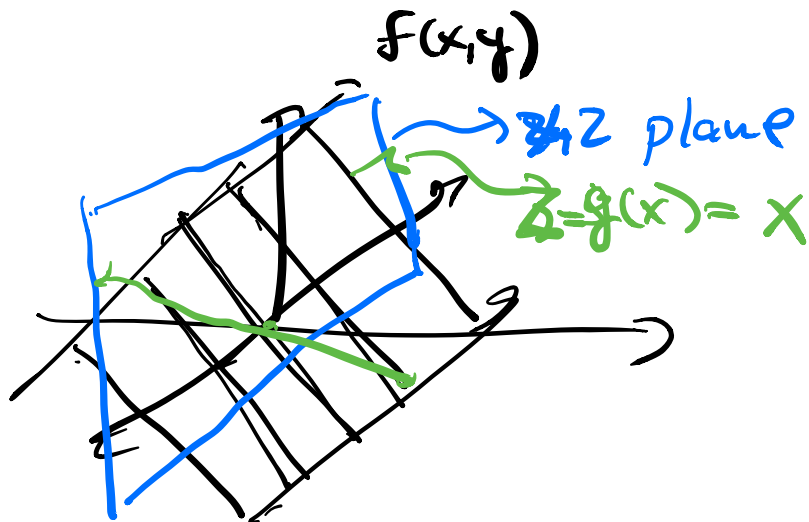
$$\begin{aligned} z=0 & 0 = x - y & y = x \\ z=1 & 1 = x - y & y = x - 1 \\ z=2 & & y = x - 2 \\ z=-1 & & y = x + 1 \\ z=-2 & & y = x + 2 \end{aligned}$$



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 $g(x) = f(x, y_0) = x - y_0$ is a function of one variable.

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$$g'(x) = 1.$$

If $x = x_0$ is constant, $h(y) = f(x_0, y) = x_0 - y$ is a function of one variable. $\rightsquigarrow h'(y) = -1$.

Definition

Let D be a domain in \mathbb{R}^2 (or \mathbb{R}^3), $f : D \rightarrow \mathbb{R}$ a function. Let (x_0, y_0) be a point in D .

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$$\lim_{h \rightarrow 0} \frac{f(\underbrace{x_0 + h}_x, \underbrace{y_0}_h) - f(\underbrace{x_0}_x, \underbrace{y_0}_h)}{h}$$

lim $\frac{f(x+h) - f(x)}{h}$ as $h \rightarrow 0$

exists, we call it the *partial derivative with respect to x*.

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Similarly, if

$$\lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

exists, call it *partial derivative with respect to y* and write $f_y(x_0, y_0)$ or $\frac{\partial f}{\partial y}(x_0, y_0)$.

Example

Let $f(x, y) = x^2 + 4y^2$. Compute $f_x(2, -3)$ and $f_y(2, -3)$ using the limit formula.

$$\lim_{h \rightarrow 0} \frac{f(2+h, -3) - f(2, -3)}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{(2+h)^2 + 4(-3)^2 - [2^2 + 4(-3)^2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \rightarrow 0} \frac{4h + h^2}{h}$$

$$= \lim_{h \rightarrow 0} 4 + h \overset{0}{\rightarrow} = 4.$$

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Function makes sense everywhere. Compute f_x :

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(2+h, -3) - f(2, -3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{((2+h)^2 + 4(-3)^2) - (2^2 + 4(-3)^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(4 + 4h + h^2) - (4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4h + h^2}{h} \\ &= \lim_{h \rightarrow 0} (4 + h) \\ &= 4. \end{aligned}$$

Similarly,

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(2, \underline{-3+h}) - f(2, \underline{-3})}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2^2 + 4(-3+h)^2) - (2^2 + 4(-3)^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(-6)h + h^2}{h} = \lim_{h \rightarrow 0} -24 + h \overset{\text{OK}}{\nearrow} \\ &= -24 \end{aligned}$$

Exercise

Let $f(x, y) = xy^2$. Compute $f_x(2, 5)$ and $f_y(3, -1)$ using the limit formula.

$$f_x(2, 5) = \lim_{h \rightarrow 0} \frac{f(2+h, 5) - f(2, 5)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(2+h)5^2 - (2 \cdot 5^2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{50 + 25h - 50}{h} = \lim_{h \rightarrow 0} \frac{25h}{h} = 25$$

$$f_x = \frac{d}{dx} (x (\text{constant})^2) = (\text{constant})^2 = y^2$$

$$f_x(2, 5) = 5^2 = 25$$

Exercise

Let $f(x, y) = xy^2$. Compute $f_x(2, 5)$ and $f_y(3, -1)$ using the limit formula.

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(2 + h, 5) - f(2, 5)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2 + h)5^2 - 2 \cdot 5^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{25h}{h} \\ &= 25. \end{aligned}$$

Continued

$$f(x, y) = xy^2 \quad f_y(3, -1)$$

$$\lim_{h \rightarrow 0} \frac{f(3, -1+h) - f(3, -1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3(-1+h)^2 - (3 \cdot (-1)^2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3(1 - 2h + h^2) - 3 \cdot 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-6h + 3h^2}{h} = \lim_{h \rightarrow 0} -6 + 3h = -6$$

$$\frac{\partial}{\partial y} f(x, y) = \frac{d}{dy} ((\text{constant})y^2) = 2(\text{constant})y = 2xy$$

$$f_y(3, -1) = 2(3)(-1) = -6$$

Similarly,

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(3, -1 + h) - f(3, -1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3(-1 + h)^2 - 3(-1)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{-6h}{h} \\ &= -6. \end{aligned}$$

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Example: Let $f(x, y) = x^2 + 4y^2$. Compute $f_x(x, y)$ and $f_y(x, y)$.

$$f_x(x, y) = \frac{d}{dx} \underbrace{g(x)}_{\text{constant}} = \frac{d}{dx} (x^2 + \text{constant}) = 2x$$

$$f_y(x, y) = \frac{d}{dy} (h(y)) = \frac{d}{dy} (\text{constant} + 4y^2) = 8y$$

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Example: Let $f(x, y) = x^2 + 4y^2$. Compute $f_x(x, y)$ and $f_y(x, y)$. If y is frozen, this looks like a function of only x :
 $g(x) = x^2 + \text{constant}$.

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Double check with previous computation: $f_x(2, -3) = 2(2) = 4$, and $f_y(2, -3) = 8(-3) = -24$.

More examples

Example: Let $f(x, y) = x^3 y^2$.

$$\frac{\partial f}{\partial x}(x, y) = \frac{d}{dx}(g(x)) \quad \begin{array}{l} \underline{y=c} \\ \underline{f} \end{array} = \frac{d}{dx}(c^2 x^3) = 3c^2 x^2 = 3y^2 x^2$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{d}{dy}(h(y)) \quad \begin{array}{l} x=c \\ \underline{f} \end{array} = \frac{d}{dy}(c^3 y^2) = 2c^3 y = 2x^3 y$$

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Example: Let $f(x, y) = x^3y^2$. If y is a constant, this looks like $g(x) = (\text{constant})x^3$, which has derivative $3(\text{constant})x^2$.

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$$g'(x) = f_x(x, y) = 3y^2x^2.$$

Similarly, if x is constant, looks like $h(y) = (\text{constant})y^2$, so

$$h'(y) = f_y(x, y) = 2x^3y$$

Exercise

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Let $f(x, y) = e^{xy}$. Compute f_x and f_y .

$$f_x = \frac{d}{dx} (e^{x(\text{constant})}) = e^{x(\text{constant})} (\text{constant}) = ye^{xy}$$

$$f_y = \frac{d}{dy} (e^{\text{constant} y}) = e^{\text{constant} y} (\text{constant}) = xe^{xy}$$

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$$f_y(x, y) = xe^{xy}$$

Exercises

Exercise: Compute both partial derivatives of the function

$$f(x, y) = \underline{xy} - \underline{\cos(x)}. \quad \sin(x) \quad \sin x$$

$$f_x(x, y) = y + \sin(x)$$

$$f_y(x, y) = \underline{x}$$

Exercise: Compute both partial derivatives of the function $f(x, y) = xy - \cos(x)$.

$$f_x(x, y) = y + \sin(x), \quad f_y(x, y) = x$$

Skip

Exercise: Compute both partial derivatives of the function

$$f(x, y) = \cos(xy)$$

Exercise: Compute both partial derivatives of the function
 $f(x, y) = \cos(xy)$

$$f_x(x, y) = -y \sin(xy), \quad f_y(x, y) = -x \sin(xy)$$

Exercise: Compute both partial derivatives of the function

$$f(x, y) = e^{x^2y} \cos(x) + \sin^2(y^2).$$

product rule

$(\sin(y^2))^2$
double chain rule

$$\partial_x (f(x, y) g(x, y)) = (\partial_x f) g + f (\partial_x g)$$

$$\partial_x (f + g) = \partial_x f + \partial_x g$$

$$\partial_x f = \partial_x (e^{x^2y} \cos(x) + \sin^2(y^2))$$

$$= \partial_x (e^{x^2y} \cos(x)) + \partial_x (\sin^2(y^2))$$

$$= \partial_x (e^{x^2y}) \cos(x) + e^{x^2y} \partial_x (\cos(x))$$

$$= e^{x^2y} (2xy) \cos(x) - e^{x^2y} \sin(x)$$

$$\partial_y f = e^{x^2y} (x^2) \cos(x) + 2(\sin(y^2)) \cos(y^2) \partial_y$$

$$= x^2 e^{x^2y} \cos(x) + 4y \sin(y^2) \cos(y^2)$$

Exercise: Compute both partial derivatives of the function $f(x, y) = e^{x^2y} \cos(x) + \sin^2(y^2)$.

$$f_x(x, y) = 2xye^{x^2y} \cos(x) - e^{x^2y} \sin(x)$$

and

$$f_y(x, y) = x^2 e^{x^2y} \cos(x) + 2 \sin(y^2) \cos(y^2) 2y$$

Exercise

Exercise: Let $G(r, s, t) = t^2 s - 14rst$. Compute all three partial derivatives.

$$\partial_r G = \frac{d}{dr} \left[\underbrace{(\text{constant } 2)^2}_{\text{constant } 2} (\text{constant } 2) - \underbrace{14r}_{\text{constant } 2} (\text{constant } 2) \right]$$
$$= -14st$$

$$\partial_s G = t^2 - 14rt$$

$$\frac{d}{dt} t^n = n(t^{n-1})$$

$$\partial_t G = \underline{2}ts - 14rs$$

Exercise: Let $G(r, s, t) = t^2s - 14rst$. Compute all three partial derivatives.

$$G_r(r, s, t) = -14st, \quad G_s(r, s, t) = t^2 - 14rt, \quad G_t(r, s, t) = 2ts - 14rs$$

Higher order derivatives

Recall: in Calc I, if $u(x)$ is differentiable and u' is differentiable, then $u''(x)$ is computed by $(u'(x))'$.

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Recall: in Calc I, if $u(x)$ is differentiable and u' is differentiable, then $u''(x)$ is computed by $(u'(x))'$. If $f(x, y)$ is a function of 2 variables, there are *four* second derivatives: If $f(x, y)$ is differentiable in x and $f_x(x, y)$ is differentiable in x ,

$$f_{xx}(x, y) = \frac{\partial f_x}{\partial x}(x, y)$$

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$$f_{\underline{xx}}(x, y) = \frac{\partial f_x}{\partial x}(x, y)$$

Similarly

$$\begin{aligned} f_{\underline{xy}} &= \frac{\partial f_x}{\partial y}(x, y) \\ \left\langle \right. & \\ f_{\underline{yx}} &= \frac{\partial f_y}{\partial x}(x, y) \end{aligned}$$

$$f_{\underline{yy}}(x, y) = \frac{\partial f_y}{\partial y}(x, y)$$

Examples

Let $f(x, y) = e^{xy}$. Then $f_x = ye^{xy}$ and $f_y = xe^{xy}$, so

$$f_{xx} = \frac{\partial f_x}{\partial x} = \frac{\partial}{\partial x} (ye^{xy}) = ye^{xy} (y) = y^2 e^{xy}$$

$$f_{xy} = \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y} (ye^{xy}) = e^{xy} + y(e^{xy} x) = e^{xy} + xy e^{xy}$$

$$f_{yx} = \frac{\partial f_y}{\partial x} = \frac{\partial}{\partial x} (xe^{xy}) = e^{xy} + x(e^{xy} \cdot y) = e^{xy} + xy e^{xy}$$

$$f_{yx} = f_{xy} \quad f_{yy} = \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial y} (xe^{xy}) = x^2 e^{xy}$$

Examples

Let $f(x, y) = e^{xy}$. Then $f_x = ye^{xy}$ and $f_y = xe^{xy}$, so

$$\frac{\partial f_x}{\partial x} = \frac{\partial(\text{const})e^{(\text{const})x}}{\partial x} = (\text{const})^2 e^{(\text{const})x} = y^2 e^{xy}$$

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For f_{xy} , need to use product rule:

$$\frac{\partial f_x}{\partial y} = \frac{\partial ye^{(\text{const})y}}{\partial y} = e^{(\text{const})y} + y(\text{const})e^{(\text{const})y} = e^{xy} + xye^{xy}$$

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Similarly, $f_{yx} = e^{xy} + xye^{xy}$ and $f_{yy} = x^2 e^{xy}$.

Exercises

$$f_x = -\sin(xy)(y) = -y \sin(xy)$$

$$f_y = -\sin(xy)(x) = -x \sin(xy)$$

Exercise: Compute all second order partial derivatives of the function $f(x, y) = \cos(xy)$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(-y \sin(xy))$$

$$= -\sin(xy) - y(\cos(xy)x) = -\sin(xy) - xy \cos(xy)$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(-x \sin(xy))$$

$$= -\sin(xy) - x \cos(xy)y = -\sin(xy) - xy \cos(xy)$$

Exercise: Compute all second order partial derivatives of the function $f(x, y) = \cos(xy)$

$$f_x(x, y) = -y \sin(xy), \quad f_y(x, y) = -x \sin(xy)$$

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so

$$f_{xx} = -y^2 \cos(xy), \quad f_{xy} = -y \sin(xy) + xy \cos(xy)$$

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so

$$f_{xx} = -y^2 \cos(xy), \quad f_{xy} = \cancel{-y \sin(xy)} + \cancel{xy \cos(xy)}$$

and

$$f_{yx} = \underline{-\sin(xy) - xy \cos(xy)}, \quad f_{yy} = -x^2 \cos(xy)$$