Chain Rule

June 24, 2020

The chain rule in 1-d

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Definition/Theorem: Let D be a domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, $f : D \to \mathbb{R}$. If all first-order partial derivatives exist and are continuous, we say f is differentiable.

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \langle \partial_x f, \partial_y f \rangle \cdot \vec{u}$

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Chain rule for 1 independent variable

Theorem: Let D be a domain in $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a differentiable function. Let $x(t)$ and $y(t)$ describe a differentiable parametrized curve in D.

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$$g'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t).$$

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Notation: Some times written as: if $z = f(x(t), y(t))$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

$z = f(x(t), y(t), w(t))$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial w}\frac{dw}{dt}$$

level set are lines with slope -1

$y = 3(-x) \rightarrow$ line through origin with slope -3

Let $f(x, y) = x + y$, $x(t) = -t$, $y(t) = 3t$. Sketch $(x(t), y(t))$ and some level sets for f. Let $z = f(x(t), y(t)) = x(t) + y(t)$. Find dz/dt.

$t=0$ $x=0$

$t=1$ $x=-1$

$x = -t$

$-x = t$

$y(t) = 3t = 3(-x)$

$= -3x$

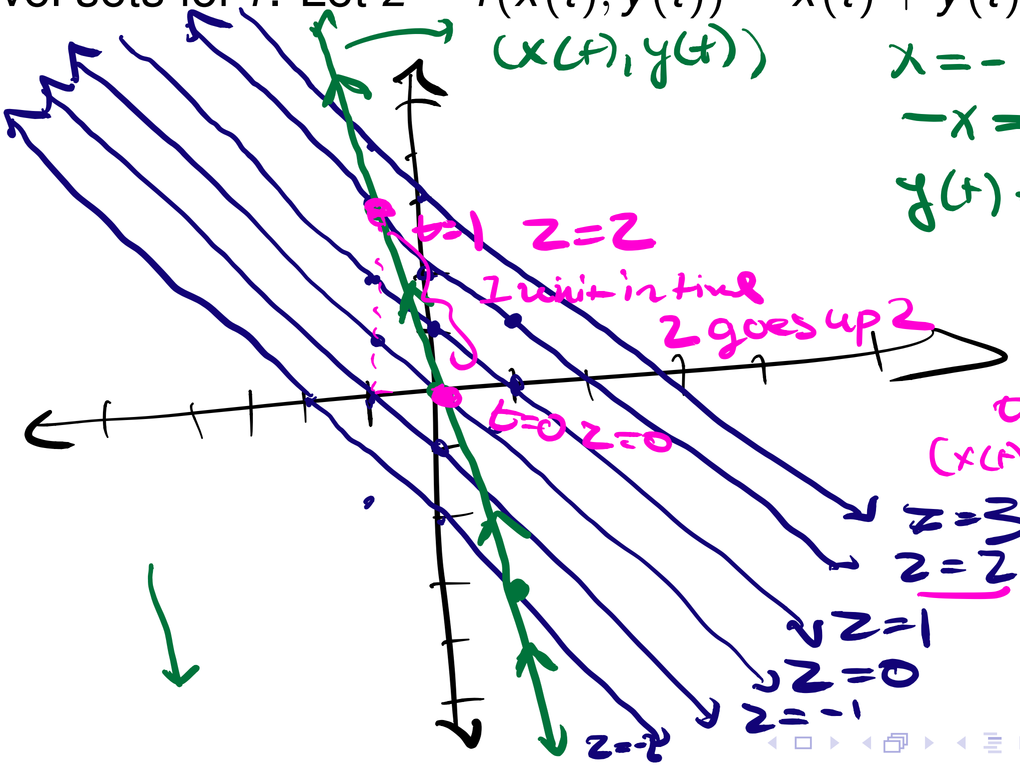
$z=0$

$z = x + y$

$y = 2 - x$

$z=1$

$y = 1 - x$



$t=1$

$(x(t), y(t)) = (-1, 3)$

$\frac{dz}{dt} = 2$

Example: linear f

Let $f(x, y) = x + y$, $x(t) = -t$, $y(t) = 3t$. Sketch $(x(t), y(t))$ and some level sets for f. Let $z = f(x(t), y(t)) = x(t) + y(t)$. Find dz/dt.

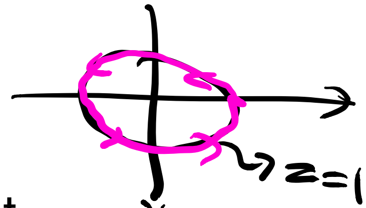
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (1)(-1) + (1)(3)$$

$$= (1)(-1) + (1)(3) = 2.$$

Note: *Direction* of the curve $(x(t), y(t))$ matters!

$(0,-1)$ $\quad 1 = x^2 + y^2$

$f(0,-1) = 1$



Let $f(x, y) = x^2 + y^2$, $x(t) = \cos t$, $y(t) = \sin(t)$. Let $z = f(x(t), y(t))$. Find dz/dt at the point $t = -\pi/2$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= 2x(-\sin t) + 2y\cos(t)$$

$$= 2(0)(-\sin(-\pi/2)) + 2(-1)\cos(-\pi/2)$$

$$= 0.$$

$x(-\frac{\pi}{2}) = \cos(-\frac{\pi}{2}) = 0$

$y(-\pi/2) = \sin(-\frac{\pi}{2}) = -1$

$\nabla f = \langle 2x, 2y \rangle$

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

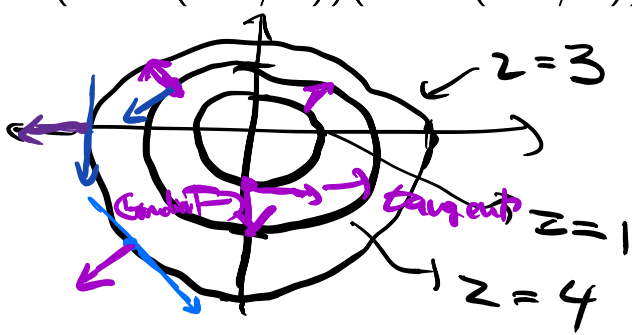
$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$\nabla f(0,-1) = \langle 0, -2 \rangle$ $\quad \langle 0, -1 \rangle$ $\quad \vec{r}\,'(-\pi/2) = \langle 1, 0 \rangle$

Let $f(x, y) = x^2 + y^2$, $x(t) = \cos t$, $y(t) = \sin(t)$. Let $z = f(x(t), y(t))$. Find dz/dt at the point $t = -\pi/2$.

$\nabla f \cdot \vec{r}\,' = \langle 0, -2 \rangle \cdot \langle 1, 0 \rangle = 0$

$$
\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (2x)(-\sin(t)) + (2y)(\cos(t)) \\
&= (2\cos(-\pi/2))(-\sin(-\pi/2)) + (2\sin(-\pi/2))(\cos(\pi/2)) = 0
\end{aligned}
$$



$D_{\vec{u}} f = \nabla f \cdot \vec{u} = 0$

Continued

Continued

$x(-\pi/2) = 0 \quad y(-\pi/2) = -e^{-3\pi/2}$

Continue to let $f = x^2 + y^2$. Now let $x(t) = e^{3t}\cos(t)$, $\quad e^{-3\pi/2}\cos(-\frac{\pi}{2})$, 0
$y(t) = e^{3t}\sin(t)$, $z = f(x(t), y(t))$. Find dz/dt at the point $t = -\pi/2$.

$$\frac{dx}{dt} = 3e^{3t}\cos(t) - e^{3t}\sin(t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{dy}{dt} = 3e^{3t}\sin(t) + e^{3t}\cos(t)$$

→ only nonzero part

$$\frac{dz}{dt} = (2x)\left(3e^{3t}\cos(t) - e^{3t}\sin(t)\right) + 2y\left(3e^{3t}\sin t + e^{3t}\cos t\right)$$

0, 0, $\neq 0$, 0, 0

$$= 2(-e^{-3\pi/2})\left(3e^{-3\pi/2}\sin(-\pi/2)\right)$$

-1

$$= 6e^{-3\pi/2} \cdot e^{-3\pi/2} = 6e^{-3\pi}.$$

Continued

Continue to let $f = x^2 + y^2$. Now let $x(t) = e^{3t}\cos(t)$, $y(t) = e^{3t}\sin(t)$, $z = f(x(t), y(t))$. Find dz/dt at the point $t = -\pi/2$.

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (2x)(3e^{3t}\cos(t) - e^{3t}\sin(t)) + (2y)(3e^{3t}\sin(t) + e^{3t}\cos(t)) \\
&= (2e^{-3\pi/2}\cos(-3\pi/2))(3e^{-3\pi/2}\cos(-3\pi/2) - e^{-3\pi/2}\sin(-3\pi/2)) \\
&\quad + (2e^{-3\pi/2}\sin(-\pi/2)(3e^{-3\pi/2}\sin(-3\pi/2))(3e^{-3\pi/2}\sin(-3\pi/2) \\
&\quad + e^{-3\pi/2}\cos(-3\pi/2)) \\
&= 2e^{-3\pi/2}(3e^{-3\pi/2}) = 6e^{-6\pi/2}.
\end{aligned}$$

$$\frac{\partial z}{\partial x} = ye^{xy} \qquad \frac{dx}{dt} = 3t^2$$

$$\frac{\partial z}{\partial y} = xe^{xy} + 2y \qquad \frac{dy}{dt} = 4t^3 - 9$$

Let $z = e^{xy} + y^2$, $x = 1 + t^3$, and $y = t^4 - 9t$. Compute dz/dt.

$$\frac{dz(t)}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= ye^{xy}(3t^2) + (xe^{xy} + 2y)(4t^3 - 9)$$

$$= (t^4 - 9t)e^{(1+t^3)(t^4-9t)}(3t^2) + \big((1+t^3)e^{(1+t^3)(t^4-9t)} + 2(t^4 - 9t)\big)(4t^3 - 9)$$

Example

Let $z = e^{xy} + y^2$, $x = 1 + t^3$, and $y = t^4 - 9t$. Compute dz/dt.

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (ye^{xy})(3t^2) + (xe^{xy} + 2y)(4t^3 - 9) \\
&= ((t^4 - 9t)e^{(1+t^3)(t^4-9t)}(3t^2) \\
&\quad + ((1 + t^3)e^{(1+t^3)(t^4-9t)} + 2(t^4 - 9t))(4t^3 - 9)
\end{aligned}$$

$$\frac{\partial z}{\partial x} = 2xy\cos(x^2y) - 1 \qquad y = \sin(2t)$$

$$\frac{\partial z}{\partial y} = x^2\cos(x^2y) \qquad \frac{dx}{dt} = -2t \qquad \frac{dy}{dt} = 2\cos(2t)$$

Let $z = \sin(x^2y) - x$, $x = 1 - t^2$, and $y = \sin(2t)$. Compute dz/dt.

$$\sin((1-t^2)^2\sin(2t))$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= [2xy\cos(x^2y) - 1][-2t] + [x^2\cos(x^2y)][2\cos(2t)]$$

$$= [2(1-t^2)\sin(2t)\cos((1-t^2)^2\sin(2t)) - 1][-2t]$$

$$+ [(1-t^2)^2\cos((1-t^2)^2\sin(2t))]\,2\cos(2t)$$

Exercise

Let $z = \sin(x^2 y) - x$, $x = 1 - t^2$, and $y = \sin(2t)$. Compute dz/dt.

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (2xy\cos(x^2 y) - 1)(-2t) + (x^2\cos(x^2 y))(2\cos(2t)) \\
&= (2(1-t^2)(\sin(2t)\cos((1-t^2)^2\sin(2t))))(-2t) \\
&\quad + ((1-t^2)^2\cos((1-t^2)^2\sin(2t))(2\cos(2t).
\end{aligned}$$

$$\frac{dx}{dt} = 2\cos(t)(-\sin t)$$

$$\frac{dy}{dt} = (1+t^2)^{-3/2}(-1/2)(2t)$$

$$x(\pi) = \cos^2(\pi) = 1$$

$$y(\pi) = (1+\pi^2)^{-1/2}$$

Let $z = 2xy - x^2 + \cos(xy^2)$, $x = \cos^2(t)$, and $y = (1+t^2)^{-1/2}$ at the point $t = \pi$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = 2y - 2x + y^2(-\sin(xy^2))$$

$$\frac{\partial z}{\partial y} = 2x + 2xy(-\sin(xy^2))$$

$-\sin(\pi) = 0$

$$= [2y - 2x + y^2(-\sin(xy^2))](2\cos t(-\sin t))$$ → First term is 0.

$$+ [2x + 2xy(-\sin(xy^2))](-t(1+t^2)^{-3/2})$$

$$= [2 - 2(1+\pi^2)^{-1/2}\sin((1+\pi^2)^{-1})][-\pi(1+\pi^2)^{-3/2}]$$

Example

Let $z = 2xy - x^2 + \cos(xy^2)$, $x = \cos^2(t)$, and $y = (1 + t^2)^{-1/2}$ at the point $t = \pi$.
At $t = \pi$, $x = 1$ and $y = (1 + \pi^2)^{-1/2}$, so

$$\begin{aligned}
\frac{dz}{dt} &= (2y - 2x - y^2 \sin(xy^2))(-2\sin(t)\cos(t)) \\
&\quad + (2x - 2xy\sin(xy^2))(-t(1 + t^2)^{-3/2}) \\
&= (2(1 + \pi^2)^{-\frac{1}{2}} - 2 - (1 + \pi^2)^{-1}\sin((1 + \pi^2)^{-1}))(0) \\
&\quad + (2 - 2(1 + \pi^2)^{-1/2}\sin((1 + \pi^2)^{-1})(-\pi(1 + \pi^2)^{-3/2}) \\
&= (2 - 2(1 + \pi^2)^{-1/2}\sin((1 + \pi^2)^{-1}))(-\pi(1 + \pi^2)^{-3/2})
\end{aligned}$$

$\frac{\partial w}{\partial x} = y - 2xyz^3 \quad \frac{\partial w}{\partial z} = 4z - 3x^2yz^2 \quad \frac{dy}{dt} = 2t$

$\frac{\partial w}{\partial y} = x - x^2z^3 \quad \frac{dx}{dt} = -\pi \sin(\pi t) \quad \frac{dz}{dt} = 6t^2$

Let $w = xy + 2z^2 - x^2yz^3$, $x = \cos(\pi t)$, $y = t^2 - 1$, and $z = 2t^3$.
Find dw/dt when $t = 1$. $x(1) = -1 \quad y(1) = 0, \quad z(1) = 2$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= (y - 2xyz^3)(-\pi \sin(\pi t)) + (x - x^2z^3)(2t) + (4z - 3x^2yz^2)(6t^2)$$

$$= (-1 - (-1)^2(2)^3)(2) + (4(2) - 0)(6)$$

$$= (-1 - 8)(2) + (8)(6)$$

$$= -18 + 48 = \boxed{30}$$

Exercise

Let $w = xy + 2z^2 - x^2yz^3$, $x = \cos(\pi t)$, $y = t^2 - 1$, and $z = 2t^3$.
Find dw/dt when $t = 1$.
At $t = 1$, $x = -1$, $y = 0$, and $z = 2$, so

$$\frac{dw}{dt} = (y - 2xyz^3)(-\pi \sin(\pi(t))) + (x - x^2z^3)(2t)$$
$$+ (4z - 3x^2yz^2)(6t^2)$$
$$= (0) + (-3)(2) + (8 - 0)(6)$$
$$= 42.$$

Chain rule with two independent variables

This gets complicated...

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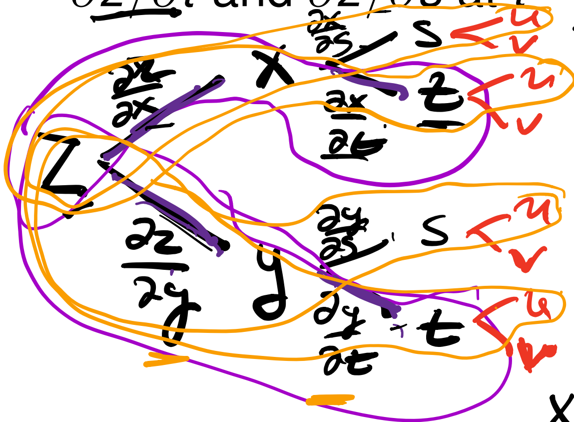
Example: Let $z = x^2y + 2y^2 - 6$, $x = ts$, $y = t^2 + s$. Compute $\partial z/\partial t$ and $\partial z/\partial s$ at $t = 1$, $s = -2$.

$x(1,-2) = -2$



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$x(1,-2) = -2$, $y(1,-2) = -1$, $t=1$, $s=-2$

$$\frac{\partial z}{\partial x} = 2xy \qquad \frac{\partial x}{\partial t} = s \qquad \frac{\partial y}{\partial t} = 2t$$

$$\frac{\partial z}{\partial y} = x^2 + 4y \qquad \frac{\partial x}{\partial s} = t \qquad \frac{\partial y}{\partial s} = 1$$

$$\frac{\partial z}{\partial t} = 2xys + (x^2+4y)(2t)$$

$$= \underline{2(-2)(-1)(-2)} + (\underbrace{(-2)^2+4(-1)}_{0})(2\cdot 1)$$

$$= -8$$

$$\frac{\partial z}{\partial s} = 2\underline{xyt} + (\underbrace{x^2+4y}_{0})(1)$$

$$= 2(-2)(-1)(1) = 4$$

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At $t = 1$, $s = -2$, $x = -2$, $y = -1$, so

$$\begin{aligned}\frac{\partial z}{\partial t} &= (2xy)(s) + (x^2 + 4y)(2t) \\ &= (4)(-2) + (4 - 4)(2) = -8\end{aligned}$$

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$$\begin{aligned}\frac{\partial z}{\partial s} &= (2xy)(t) + (x^2 + 4y)(1) \\ &= (4)(1) + (4 - 4)(1) = 4.\end{aligned}$$