

LOCAL SMOOTHING FOR THE SCHRÖDINGER EQUATION ON A MULTI-WARPED PRODUCT MANIFOLD WITH INFLECTION-TRANSMISSION TRAPPING

HANS CHRISTIANSON AND DERRICK NOWAK

ABSTRACT. Geodesic trapping is an obstruction to dispersive estimates for solutions to the Schrödinger equation. Surprisingly little is known about solutions to the Schrödinger equation on manifolds with degenerate trapping, since the conditions for degenerate trapping are not stable under perturbations. In this paper we extend some of the results of [CM14] on inflection-transmission type trapping on warped product manifolds to the case of *multi-warped* products. The main result is that the trapping on one cross section does not interact with the trapping on other cross sections provided the manifold has only one infinite end and only inflection-transmission type trapping.

1. INTRODUCTION

In this paper, we study the effects of inflection-transmission type trapping on local smoothing estimates for solutions to the Schrödinger equation on a multi-warped product manifold. Inflection-transmission trapping on a warped product manifold was introduced in [CM14] by Christianson-Metcalfe as a semi-stable type of trapping. The warped product structure allows the authors to separate variables and study an essentially one-dimensional problem. The purpose of this paper is to continue that study into the context of a multi-warped product manifold where the trapping can occur on different cross sections. This breaks the symmetry of the single warped product manifold so that the problem is no longer a one-dimensional problem.

1.1. Multi-warped product manifold. The most familiar example of a *warped* product manifold is a surface of revolution, which involves a defining curve revolved around a line. This means the defining curve is warping the circle at each point to change the radius along the surface. The second most familiar warped product manifold is \mathbb{R}^n in polar coordinates. That is, $\mathbb{R}^n = \mathbb{R}_+ \times \mathbb{S}^{n-1}$ together with the metric

$$g = dx^2 + x^2 g_{\mathbb{S}^{n-1}}.$$

Here we refer to $A(x) = x$ as the “warping” function. Let $A(x) : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a smooth function satisfying $A(x) > 0$ for $x > 0$ and $A(x) \sim x$ near $x = 0$ and outside a compact set. Let M be a compact Riemannian manifold without boundary. Then $X = \mathbb{R}_+ \times M$ with the metric

$$g = dx^2 + A^2(x)g_M$$

is called a warped product with cross section M and warping function $A(x)$. It is “Euclidean” outside a compact set because $A(x) = x$ outside a compact set, and it has one “infinite end” since we are only working with $x \in \mathbb{R}_+$ and $A(x) = x$ near $x = 0$.

A multi-warped product is a product of two or more cross section manifolds warped by different warping functions. We will assume our manifold is Euclidean outside a compact set so that infinity looks like a compact product manifold warped in the usual polar coordinates. In this paper, we will specialize to the case with only one infinite end.

A multi-warped product manifold is defined as follows: Let M_1, M_2, \dots, M_N be compact Riemannian manifolds without boundary. Denote the corresponding metrics g_{M_1}, \dots, g_{M_N} , and suppose they have dimensions n_1, \dots, n_N respectively. Let $A_1, \dots, A_N : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfy $A_j(x) > 0$, $A_j(x) = x$ near $x = 0$ and outside a compact set. Let

$$X = \mathbb{R}_+ \times M_1 \times M_2 \times \dots \times M_N$$

Date: March 30, 2021.

with the metric

$$g = dx^2 + A_1(x)^2 g_{M_1} + \dots + A_N(x)^2 g_{M_N}$$

Then X is a multi-warped product manifold with cross sections M_1, \dots, M_N . It is Euclidean at infinity, since the metric is

$$g = dx^2 + x^2(g_{M_1} + \dots + g_{M_N})$$

for x outside a compact set. The metric g takes the same form in a neighborhood of $x = 0$, so X is Euclidean near 0 as well. Observe that the dimension of X is $n_1 + n_2 + \dots + n_N + 1$.

Many of these assumptions about the geometry can be relaxed in various ways without significantly changing the analysis in this paper. It is also possible to study multi-warped product manifolds with two ends, which just means the $A_j(x)$ are positive functions on \mathbb{R} which equal $|x|$ outside a compact set. We will study the Schrödinger equation on such manifolds in a subsequent paper.

2. STATEMENT OF RESULTS

Let X be a Riemannian manifold with metric g , and let $-\Delta_g$ denote the corresponding Laplace-Beltrami operator. The Schrödinger equation on X is

$$(2.1) \quad \begin{cases} (D_t - \Delta_g)u(t, x) = 0 \text{ on } \mathbb{R}_t \times X, \\ u(0, x) = u_0(x), \end{cases}$$

where u_0 is in some reasonable Sobolev space. Here we use the convention $D_t = \frac{1}{i}\partial_t$. Our goal is to understand how the geometry of X affects solutions to (2.1). In the following subsection we construct a multi-warped product manifold with inflection-transmission type trapping.

2.1. Construction of the Manifold. In order to make the present paper as clear as possible, we specialize to the case where there are only two cross sections, both circles.

We consider smooth functions A_1, A_2 and constants $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$ such that for $j = 1, 2$, $A_j(x) = x$ for x near 0 and outside a compact set, $A_j(x) > 0$ for $x > 0$, $A'_j(x) \geq 0$,

$$A_1^2(x) = \begin{cases} C_1(x-1)^{2m_1+1} + C_2, & x \sim 1 \\ \frac{1}{C_3-C_4x}, & x \sim 2 \end{cases}$$

and

$$A_2^2(x) = \begin{cases} \frac{1}{C_5-C_6x}, & x \sim 1 \\ C_7(x-2)^{2m_2+1} + C_8, & x \sim 2 \end{cases}$$

where $A'_1(x) = 0$ if and only if $x = 1$ and $A'_2(x) = 0$ if and only if $x = 2$. Here m_1 and m_2 are positive integers. The constants are needed to make sure such functions exist while maintaining that A_2^2, A_1^2 have only one point where the derivative is 0. We are also assuming that $A_1^{-2}(x)$ is linear and decreasing near $x = 2$ and $A_2^{-2}(x)$ is linear and decreasing near $x = 1$. A sketch of A_1 and A_2 are found in Figure 1.

Now let $X = \mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{S}^1$ be a half line crossed with two circles. Let θ and ω parametrize the circles, and let

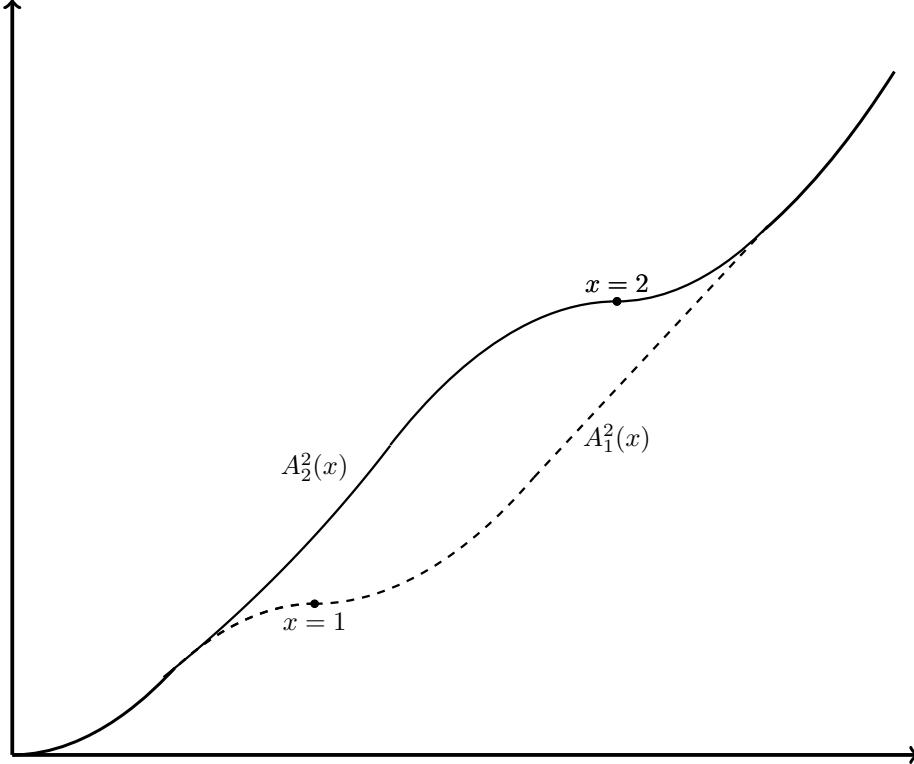
$$g = dx^2 + A_1^2(x)d\theta^2 + A_2^2(x)d\omega^2,$$

making X a multi-warped product manifold.

Theorem 2.2. *Let (X, g) be the multi-warped product constructed above. Suppose u solves (2.1) on X with $u_0 \in \mathcal{S}(X)$. Let $m = \max(m_1, m_2)$. Then for each $T > 0$ there exists a constant C such that*

$$(2.3) \quad \int_0^T \|\langle x \rangle^{-3/2} u\|_{H^1(X)}^2 dt \leq C \|u_0\|_{H^{\frac{2m+1}{2m+3}}(X)}^2$$

Remark 2.4. Due to the quasimode construction in [CM14] the estimate in (2.3) is sharp.

FIGURE 1. The functions A_1 and A_2 .

Remark 2.5. The estimate (2.3) expresses that locally in space and on average in time the solution u is $2/(2m+3)$ derivatives smoother than the initial data. Because of this, estimate (2.3) is called a local smoothing estimate. See Subsection 2.2 for motivation and history of local smoothing type estimates.

Remark 2.6. Additionally, we state the result in the form listed because it follows naturally from the estimates below. However, since $u_0 \in \mathcal{S}(X)$ we can commute with an appropriate pseudo-differential operator to get that for any s ,

$$\int_0^T \|\langle x \rangle^{-3/2} u\|_{H^{s+\frac{2}{2m+3}}(X)}^2 dt \leq C \|u_0\|_{H^s(X)}^2$$

Remark 2.7. The power $-3/2$ in the weight function is not optimal, but helps our computations later. We have assumed $u_0 \in \mathcal{S}$ to avoid any regularity issues, but a density argument can be used to extend this result to rougher initial data.

Remark 2.8. We again want to emphasize that A_1^2 has an inflection point of order $2m_1 + 1$ at $x = 1$, A_2^2 has an inflection point of order $2m_2 + 1$ at $x = 2$ and that A_1 and A_2 give the Euclidean metric near $x = 0$ and when x is large. We also make A_1^{-2} linear near $x = 2$ and A_2^{-2} linear near $x = 1$ to make some of the computation easier. However, we expect that this conditioned can be loosened and still give the same result.

2.2. Motivation and History. The Schrödinger equation is one of a large family of *dispersive* equations, which are equations whose solutions propagate in a way that depends on the frequency of oscillation. Dispersive equations have conserved quantities, often expressing that the mass or size of oscillations are preserved in time. For the Schrödinger equation on \mathbb{R}^n , the H^s norm of a solution is preserved in time. In other words, at any time t , the solution has the same regularity as the initial data. The local smoothing effect for solutions to the Schrödinger

equation expresses that, even though a solution to the Schrödinger equation has the same regularity as the initial data, on average in time and locally in space the solution is $1/2$ derivative smoother.

The local smoothing estimate for solutions to the Schrödinger equation on \mathbb{R}^n is that for any T and any $\varepsilon > 0$, there exists a $C > 0$ such that

$$\int_0^T \|\langle x \rangle^{-1/2-\varepsilon} e^{it\Delta} u_0\|_{H^{1/2}}^2 dt \leq C \|u_0\|_{L^2}^2.$$

This type of estimate has been studied in a number of different contexts with dispersive equations of varying orders [Sjö87, CS88, Veg98]. These studies were extended to the case of non-trapping asymptotically Euclidean manifolds in [CKS95, Doi96]. That trapping necessarily causes a loss in regularity was proved by Doi [iD96].

There have been a number of results about manifolds with trapping. If the trapping is unstable and non-degenerate, the loss in regularity is logarithmic [Bur04, Chr07, Chr08, Chr11, Dat09]. Non-degenerate trapping allows the use of quantum Birkhoff normal forms to have an invariant definition of hyperbolic trapping. If the trapping is unstable but degenerate, normal forms are not available so the examples are limited. In [CW13] the authors show there is a local smoothing estimate with sharp polynomial loss. In [CM14] the authors introduce the semi-stable inflection-transmission trapping, further studied in the present paper, and demonstrate a local smoothing estimate with sharp polynomial loss. In [Chr18], the author proves that unstable but infinitely degenerate trapping causes a complete loss.

The intuition behind the non-trapping estimates is as follows: In \mathbb{R}^n , if u_0 is sufficiently smooth, we can use the Fourier transform to write down the solution:

$$u(t, x) = c_n \int u_0(y) e^{i(-t|\xi|^2 + \xi \cdot (x-y))} dy d\xi,$$

where c_n is a dimensional constant. Restricting our attention to \mathbb{R}^2 , the solution has phase function $-t\xi^2 + \xi(x-y)$ which is stationary when $-2t\xi + (x-y) = 0$, or $x = y + 2t\xi$. This means that a solution at frequency ξ propagates at speed 2ξ . This has the effect that a solution leaves a compact set in space in time $t \sim \xi^{-1}$. Then integrating the $H^s(\mathbb{R}^2)$ norm in time gains ξ^{-1} over $|\xi|^{2s} |\hat{u}|^2$, or $1/2$ derivative on each copy of the solution u .

We also see from this heuristic that solutions propagate along geodesics in the sense that they follow straight lines as they propagate out to infinity. The same is true on manifolds, as long as all geodesics go to infinity. This is why trapping plays such an important role in local smoothing estimates. When trapping occurs, wave packets can stay coherent near the trapping which means that our \mathbb{R}^2 heuristic does not work any more, and we expect some loss in regularity.

2.3. Overview. On a warped product manifold $X = \mathbb{R}_+ \times M$ with metric $g = dx^2 + A^2(x)g_M$, the Laplacian is, up to lower order terms,

$$-\Delta = -\partial_x^2 - A^{-2}(x)\Delta_{g_M}.$$

Let $\{\varphi_j(\omega)\}$ be the orthonormal basis of $L^2(M)$ consisting of eigenfunctions:

$$-\Delta_{g_M} \varphi_j = \lambda_j^2 \varphi_j.$$

Then if $f : X \rightarrow \mathbb{C}$ is sufficiently smooth, we can separate variables:

$$f(x, \omega) = \sum f_j(x) \varphi_j(\omega),$$

so that, up to lower order terms,

$$-\Delta f = \sum (-f_j'' + A^{-2}(x)\lambda_j^2 f_j) \varphi_j.$$

On each eigenspace then one considers the operator $-\partial_x^2 + \lambda_j^2 A^{-2}(x)$. Rescaling $h = \lambda_j^{-1}$, we are led to consider the operator $P = -h^2 \partial_x^2 + V(x)$, where $V(x) = A^{-2}(x)$. The corresponding (semi-classical) symbol

is $p = \xi^2 + V(x)$. In this reduced geometry, the replacement for the geodesic flow is the Hamiltonian flow, and solutions propagate along this flow. The Hamiltonian system for this symbol is then

$$\begin{cases} \dot{x} = 2\xi, \\ \dot{\xi} = -V'(x), \\ x(0) = x_0, \\ \xi(0) = \xi_0. \end{cases}$$

If $V'(x_0) = 0$, then $(x, \xi) = (x_0, \xi_0)$ is a “trapped” solution. This corresponds to a longitudinal periodic geodesic on the original warped product.

The question of local smoothing with loss then boils down to understanding what happens to solutions of the one-dimensional semi-classical problem near critical points in phase space. This necessitates use of second microlocalization to get sharp estimates. This analysis was done in the papers [CW13] with degenerate unstable trapping, [CM14] for inflection-transmission type trapping, and in [Chr18] for infinitely degenerate critical points. The present paper is a continuation of this series of papers.

The motivation is to see how different kinds of trapping interact at different frequencies in a relatively simple geometric setting. Our main result, however, is that the trapped sets on each cross section do not see each other, so the loss in local smoothing is the same as in [CM14].

Nevertheless, there are a number of things to prove. Having a product of two compact manifolds as cross sections, one can separate variables on both cross sections. Then one is led to study a one-dimensional problem with two frequency parameters. This appears to be a complicated mess comparing different frequencies. However, we can separate variables in one cross section alone, which leaves us with a two-dimensional problem with one parameter. Since we are only separating variables in one direction, we do have to deal with derivatives in the other direction. However, a detailed microlocal frequency localization allows us to handle this problem. The fact that the trapping on one cross section does not see the trapping on the other cross section is special to the one ended case and not expected to hold in general.

3. LOCAL SMOOTHING AWAY FROM THE TRAPPING

Now that we have A_1 and A_2 defined, consider the product manifold $\mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{S}^1$ with the metric

$$g = dx^2 + A_1(x)^2 d\theta^2 + A_2(x)^2 d\omega^2.$$

Then, the laplacian is given by

$$\Delta_g = \partial_x^2 + A_1(x)^{-2} \partial_\theta^2 + A_2(x)^{-2} \partial_\omega^2 + (A'_1(x)A_1^{-1}(x) + A'_2(x)A_2^{-1}(x))\partial_x$$

Next we use a transformation to get rid of the ∂_x term. Consider the unitary transformation $T : L^2(X, dV_g) \rightarrow L^2(X, dx d\theta d\omega)$ given by

$$Tu = A_1^{1/2}(x)A_2^{1/2}(x)u$$

and set

$$\tilde{\Delta} = T\Delta_g T^{-1}.$$

This gives

$$\tilde{\Delta} = \partial_x^2 + A_1^{-2} \partial_\theta^2 + A_2^{-2}(x) \partial_\omega^2 + V(x)$$

where

$$\begin{aligned} V = & \frac{1}{4} A'_1(x)^2 A_1(x)^{-2} - \frac{1}{2} A''_1(x) A_1^{-1}(x) \\ & + \frac{1}{4} A'_2(x)^2 A_2(x)^{-2} - \frac{1}{2} A''_2(x) A_2^{-1}(x) \\ & - \frac{1}{2} A_1(x)^{-1} A_2(x)^{-1} A'_1(x) A'_2(x). \end{aligned}$$

This V is similar to the single warped product case except we have a cross term of

$$\frac{1}{2}A_1(x)^{-1}A_2(x)^{-1}A'_1(x)A'_2(x).$$

Next we want to do a positive commutator argument to get local smoothing away from $x = 1, x = 2$. Let u be a solution to $(D_t - \tilde{\Delta})u = 0$. Notice that $\tilde{\Delta}$ is of a similar form to [CW13]. Let us take $B = f(x)\partial_x$ for some general $f \in C^2(\mathbb{R})$ such that f, f', f'' are all bounded and then we will reduce to a specific case.

$$\begin{aligned} [\tilde{\Delta}, B] &= 2f'(x)\partial_x^2 + f''(x)\partial_x \\ &\quad + 2A'_1A_1^{-3}f(x)\partial_\theta^2 + 2A'_2A_2^{-3}f(x)\partial_\omega^2 + V'(x)f(x) \end{aligned}$$

Remark 3.1. Note that

$$\langle u, v \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} u\bar{v} dx d\theta d\omega.$$

and that

$$iB - (iB)^* = i[f(x), \partial_x].$$

Hence,

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{S}^1} u\overline{(f(x)D_x(D_t - \tilde{\Delta})u)} dx d\theta d\omega dt \\ &= \int_0^T \int f(x)(D_x u)\overline{(D_t - \tilde{\Delta})u} dx d\theta d\omega dt \\ &\quad + \int_0^T \int (iB - (iB)^*)u\overline{(D_t - \tilde{\Delta})u} dx d\theta d\omega dt \\ &= i\langle f(x)D_x u, u \rangle |_0^T + \int_0^T \langle (D_t - \tilde{\Delta})i^{-1}Bu, u \rangle dt. \end{aligned}$$

This follows from integrating $\partial_t(\langle f(x)D_x u, u \rangle)$ in t , using $D_t u = \tilde{\Delta}u$ and integrating by parts. It is the same computation as [CW13] and the next step in the paper follows through as well. Using the notation that $P = D_t - \tilde{\Delta}$,

$$\begin{aligned} 0 &= 2i \operatorname{Im} \int_0^T \langle i^{-1}BPu, u \rangle dt \\ &= \int_0^T \langle i^{-1}BPu, u \rangle dt - \int_0^T \langle u, i^{-1}BPu \rangle dt \\ &= \int_0^T \langle [i^{-1}B, P], u, u \rangle dt - i\langle f(x)D_x u, u \rangle |_0^T. \end{aligned}$$

Since B is independent of t this gives

$$\int_0^T \langle [B, -\tilde{\Delta}]u, u \rangle dt = -\langle f(x)D_x u, u \rangle |_0^T.$$

Let us reduce to the specific case of a function $f(x)$ given by the following: let $\zeta(x)$ be a smooth function satisfying $\zeta(x) \equiv 1$ near $x = 0$, $\zeta(x) > 0$ for all x , and $|\zeta(x)| \sim \langle x \rangle^{-3}$ for large x . Such a ζ is integrable, so let

$$f(x) = \int_0^x \zeta(t) dt.$$

Then $f(x) = x$ near $x = 0$, and there exists a constant $c > 0$ such that $f'(x) \geq c\langle x \rangle^{-3}$ for $x \geq 0$. The power -3 here is much bigger than needed, but we have chosen it so that our computation are easier. We simply are matching the power of each $A_j^{-3} \sim x^{-3}$ as $x \rightarrow \infty$.

The restriction that $f(x)$ is linear near $x = 0$ is just to maintain all the properties of Euclidean polar coordinates near $x = 0$. Integrating by parts yields

$$\begin{aligned} & \int_0^T -\langle 2f'(x)\partial_x u, \partial_x u \rangle - \langle 2A'_1 A_1^{-3} f(x)\partial_\theta u, \partial_\theta u \rangle - \langle 2A'_2 A_2^{-3} f(x)\partial_\omega u, \partial_\omega u \rangle dt \\ &= -\langle f(x)D_x u, u \rangle|_0^T + \int_0^T \langle f''(x)\partial_x u, u \rangle - \langle V'(x)f(x)u, u \rangle dt \end{aligned}$$

Let us quickly remark again that, since each $A_j(x) = x$ for x near 0 and $f(x) = x$ for x near 0, we have $A_j^{-3}(x)f(x) = x^{-2}$ near $x = 0$. We also have $V'(x) = 0$ near $x = 0$, so all terms agree with the corresponding Euclidean terms near $x = 0$.

Taking the absolute value of both sides and noting that f' , $A'_1 A_1^{-3} f(x)$, and $A'_2 A_2^{-3} f(x) \geq 0$ yields

$$\begin{aligned} & \int_0^T \|\sqrt{2f'(x)}\partial_x u\|_{L^2}^2 + \|\sqrt{2A'_1 A_1^{-3} f(x)}\partial_\theta u\|_{L^2}^2 + \|\sqrt{2A'_2 A_2^{-3} f(x)}\partial_\omega u\|_{L^2}^2 dt \\ & \leq C_1 |\langle D_x u, u \rangle|_0^T + \int_0^T C_2 |\langle \partial_x u, u \rangle| + C_3 |\langle u, u \rangle| dt \end{aligned}$$

Note that each term on the RHS is bounded by $C_T \|u_0\|_{H^{1/2}}$ for some constant C_T . Next, we want to provide lower bounds on the $\sqrt{2f'(x)}$, $\sqrt{2A'_1 A_1^{-3} f(x)}$, and $\sqrt{2A'_2 A_2^{-3} f(x)}$ terms.

First we want to bound the ∂_x^2 . Note that $f'(x) = \zeta(x)$ defined above, so there exists a positive constant $c > 0$ such that

$$\|\sqrt{2f'(x)}\partial_x u\|_{L^2}^2 \geq c \|\langle x \rangle^{-3/2} \partial_x u\|_{L^2}^2.$$

To get the correct lower bounds for the $\sqrt{2A'_1 A_1^{-3} f(x)}$, and $\sqrt{2A'_2 A_2^{-3} f(x)}$ terms we will have to estimate $A'_1 A_1^{-3}$ and $A'_2 A_2^{-3}$.

3.1. A_1 and A_2 estimates. We have that

$$A_1^2(x) \sim \begin{cases} C_1(x-1)^{2m_1+1} + C_2, & x \sim 1 \\ x^2, & x \text{ away from 1} \end{cases}$$

So, near $x = 0$

$$f(x)A'_1(x)A_1^{-3}(x) = \frac{1}{x^2} \geq C \frac{1}{x^2 \langle x \rangle^1} \geq C \frac{(x-1)^{2m_1}}{x^2 \langle x \rangle^{1+2m_1}}$$

Near $x = 1$

$$f(x)A'_1(x)A_1^{-3}(x) \sim \frac{(x-1)^{2m_1}}{(1 + (x-1)^{2m_1+1})^{3/2}} \geq C \frac{(x-1)^{2m_1}}{x^2 \langle x \rangle^{1+2m_1}}$$

When x is large

$$f(x)A'_1(x)A_1^{-3}(x) \sim \frac{1}{x^3} \geq C \frac{1}{x^2 \langle x \rangle} \geq C \frac{(x-1)^{2m_1}}{x^2 \langle x \rangle^{1+2m_1}}$$

Now just to be careful, we can consider compact sets $[\varepsilon, 1-\varepsilon]$ and $[1+\varepsilon, K]$ for K sufficiently large and ε small to handle the situation where we do not know the exact form of A_1^2 . We know that on this region $f(x), A'_1(x) > 0$ so we can find $C > 0$ sufficiently small so that

$$f(x)A'_1(x)A_1^{-3}(x) \geq C \frac{(x-1)^{2m_1}}{x^2 \langle x \rangle^{1+2m_1}} \quad x \in [\varepsilon, 1-\varepsilon] \cup [1+\varepsilon, K]$$

With $A_2(x)$, the only difference is the inflection point is at $x = 2$ and we replace m_1 with m_2 . This does not change the qualitative behavior of the estimates. We just need estimates near $x = 2$ instead of $x = 1$ and we will get $(x-2)^{2m_2}$ in the numerator instead of $(x-1)^{2m_2}$. This proves the following Lemma:

Lemma 3.2. *Let u be a solution to (2.1) on our manifold X with initial data $u_0 \in \mathcal{S}(X)$. Then for each $T > 0$, there exists a constant $C > 0$ such that*

$$\begin{aligned} (3.3) \quad & \int_0^T (\| \langle x \rangle^{-3/2} \partial_x u \|^2 + \| (x-1)^{m_1} \langle x \rangle^{-1/2-m_1} A_1^{-1} \partial_\theta u \|^2 \\ & + \| (x-2)^{m_2} \langle x \rangle^{-1/2-m_2} A_2^{-1} \partial_\omega u \|^2) dt \\ & \leq C \| u_0 \|^2_{H^{1/2}(X)}. \end{aligned}$$

Remark 3.4. The estimate (3.3) expresses that there is perfect local smoothing in the radial x direction with a loss at the trapped set on each copy of \mathbb{S}^1 . It is also clear that the statement of Theorem 2.2 could be sharpened to have loss only in θ and ω derivatives. However, we have stated the theorem in the simplest possible way to be clear.

4. SEPARATION OF VARIABLES

Consider the operator $P_1 = P_0 + V(x) = -\partial_x^2 - V_1 \partial_\theta^2 - V_2 \partial_\omega^2 - V(x)$ where $V_j = A_j^{-2}$ and $V(x)$ contains derivatives of A_j as shown above. Define a function $\varphi(x) \in C_c^\infty$ such that $0 \leq \varphi(x) \leq 1$, $\varphi(x) \equiv 1$, on $x \in [1 - \varepsilon, 1 + \varepsilon]$ for $1/4 > \varepsilon > 0$ and $\text{supp}(\varphi) \subset [1 - 2\varepsilon, 1 + 2\varepsilon]$. Since we have local smoothing away from $x = 1$ we can localize near this point. We do this now so that we can define a Fourier transform properly and do not have to worry about any integrability issues near $x = 0$ due to the metric.

Now separate one variable at a time, starting with θ . Write

$$u = \sum u_k(t, x, \omega) e^{ik\theta}, \quad u_0 = \sum u_{0,k}(x, \omega) e^{ik\theta}.$$

Then each u_k satisfies:

$$(D_t + P_k - V)((\varphi u)_k) = 2\varphi'(x) \partial_x u_k + \varphi''(x) u_k$$

where

$$P_k = -\partial_x^2 + k^2 V_1 - V_2 \partial_\omega^2.$$

Note that φ', φ'' are compactly supported away from $x = 0$ and $x = 1$.

Below we will drop the subscript k for notational purposes. Now we want to decompose the frequency into high and low angular frequency parts. The high frequency part is when the frequency in the θ direction is large compared to the frequency in the x direction. Consider an even bump function $\psi \in C_C^\infty(\mathbb{R})$ which is 1 for $|r| \leq \varepsilon$ and vanishes for $|r| \leq 2\varepsilon$ for $\varepsilon > 0$ small. Define

$$u_{\text{hi}} = \psi(D_x/k)(\varphi u), \quad u_{\text{lo}} = (1 - \psi)(\varphi u).$$

Since φ provides a cutoff near $x = 1$ and away from zero, we can define $\psi(D_x/k)$ in the usual way.

Now using the definition of u_{lo} and the fact that $D_t u = -(P_k - V)u$ we get that.

$$\begin{aligned} (D_t + P_k - V)u_{\text{lo}} &= [P_k - V, (1 - \psi)\varphi]u \\ &= (1 - \psi)[-D_x^2, \varphi]u + [k^2 V_1 - V_2 \partial_\omega^2 - V, -\psi](\varphi u) \\ &= (1 - \psi)(-2\varphi' \partial_x - \varphi'')u + [k^2 V_1 - V, -\psi](\varphi u) + [-V_2 \partial_\omega^2, -\psi](\varphi u) \\ &= (1 - \psi)(-2\varphi' \partial_x - \varphi'')u + kL_1(\varphi u) - \frac{1}{k}L_2 \partial_\omega^2(\varphi u) \end{aligned}$$

Here L_1 and L_2 are semi-classical pseudo-differential operators (with parameter $|k|^{-1}$) of order zero with wavefront set contained in $\{\psi'(\xi/k) \neq 0\} \subset \{\varepsilon \leq |\xi|/|k| \leq 2\varepsilon\}$, so we observe $|D_x| \sim |k|$ on the wavefront set of L_1 and L_2 . We will use this shortly. Now combining the above statements gives

$$(D_t + P_k - V)u_{\text{lo}} = kL_1(\varphi u) - \frac{1}{k}L_2 \partial_\omega^2(\varphi u) - (1 - \psi)(2\varphi' \partial_x u + \varphi'' u).$$

We now run the commutator argument, but insert a cutoff $\chi_1(x)$ with $\chi_1 \equiv 1$ on $\text{supp}(\varphi)$ near $x = 1$ and $\chi_1 \equiv 0$ near $x = 2$. Let us also assume that $\chi_1^{1/2}$ is still smooth. Then with $B = f(x)\partial_x$ as before, recalling that $f'(x) = \zeta(x)$,

$$(4.1) \quad \begin{aligned} & \int_0^T \langle \chi_1 [D_t + P_k - V, B] u_{\text{lo}}, u_{\text{lo}} \rangle dt \\ &= \int_0^T \langle \chi_1 (-2\zeta(x)\partial_x^2 - k^2 V'_1 f(x) + V'_2 f(x)\partial_\omega^2) u_{\text{lo}}, u_{\text{lo}} \rangle dt \\ &+ \int_0^T \langle \chi_1 (-f''(x)\partial_x + fV') u_{\text{lo}}, u_{\text{lo}} \rangle dt. \end{aligned}$$

The last line in (4.1) has only one x derivative, so is bounded as follows:

$$\left| \int_0^T \langle \chi_1 (-f''(x)\partial_x + fV') u_{\text{lo}}, u_{\text{lo}} \rangle dt \right| \leq C_T \|u_0\|_{H^{1/2}(X)}^2.$$

The ∂_ω^2 term in the second line of (4.1) is further estimated as follows: we know that for $j = 1, 2$, $V'_j(x) \leq 0$ and our function $f \geq 0$, so

$$\begin{aligned} & \int_0^T \langle \chi_1 V'_2 f(x) \partial_\omega^2 u_{\text{lo}}, u_{\text{lo}} \rangle dt \\ &= - \int_0^T \langle \chi_1 V'_2 f(x) \partial_\omega u_{\text{lo}}, \partial_\omega u_{\text{lo}} \rangle dt \\ &\geq 0. \end{aligned}$$

We also know that $-\chi_1 f V'_1 \geq 0$, so that

$$(4.2) \quad \begin{aligned} & \int_0^T \langle \chi_1 (-2\zeta(x)\partial_x^2 - k^2 V'_1 f(x) + V'_2 f(x)\partial_\omega^2) u_{\text{lo}}, u_{\text{lo}} \rangle dt \\ &\geq \int_0^T \langle -2\chi_1 \zeta(x)\partial_x^2 u_{\text{lo}}, u_{\text{lo}} \rangle dt. \end{aligned}$$

The next issue is to observe that $V'_1(1) = 0$, so does not help us eliminate the vanishing at $x = 1$ in (3.3). However, we observe that on the wavefront set of u_{lo} , we have $|k| \lesssim |D_x|$, so we want to use the Gårding inequality to estimate k in terms of D_x . Recall that $\chi_1 V'_1 f \leq 0$ and has compact support so the Gårding inequality implies there exists a constant $C > 0$ such that

$$\langle k^2 \chi_1 u_{\text{lo}}, u_{\text{lo}} \rangle \leq -C \langle \zeta(x)\partial_x^2 u_{\text{lo}}, u_{\text{lo}} \rangle + \mathcal{O}(1) \|u_{\text{lo}}\|_{H^{1/2}(X)}^2.$$

Combining this with (4.1) and (4.2)

$$\begin{aligned} & \int_0^T \langle \chi_1 (k^2 u_{\text{lo}}), u_{\text{lo}} \rangle dt \\ &\leq -C \langle \zeta(x)\partial_x^2 u_{\text{lo}}, u_{\text{lo}} \rangle + \mathcal{O}(1) \|u_{\text{lo}}\|_{H^{1/2}(X)}^2 \\ &\leq C \int_0^T \langle \chi_1 (-2\zeta(x)\partial_x^2 - k^2 V'_1 f(x) + V'_2 f(x)\partial_\omega^2) u_{\text{lo}}, u_{\text{lo}} \rangle dt + \mathcal{O}(1) \|u_{\text{lo}}\|_{H^{1/2}(X)}^2 \\ &= C \int_0^T \langle \chi_1 [D_t + P_k - V, B] u_{\text{lo}}, u_{\text{lo}} \rangle dt + \mathcal{O}_T(1) \|u_0\|_{H^{1/2}(X)}^2. \end{aligned}$$

Rearranging and using energy estimates, we have

$$(4.3) \quad \int_0^T \langle \chi_1 k u_{\text{lo}}, k u_{\text{lo}} \rangle dt \leq C_T \|u_0\|_{H^{1/2}(X)}^2 + C \left| \int_0^T \langle \chi_1 [D_t + P_k - V, B] u_{\text{lo}}, u_{\text{lo}} \rangle dt \right|.$$

Now we unpack the commutator term $\int_0^T \langle \chi_1 [D_t + P_k - V, B] u_{lo}, u_{lo} \rangle dt$. Integrating by parts yields,

$$(4.4) \quad \begin{aligned} \left| \int_0^T \langle \chi_1 [D_t + P_k - V, B] u_{lo}, u_{lo} \rangle dt \right| &\leq 2 \left| \int_0^T \langle \chi_1 B u_{lo}, (D_t + P_k - V) u_{lo} \rangle dt \right| \\ &\quad + \left| \int_0^T \langle B u_{lo}, 2\chi'_1 \partial_x u_{lo} \rangle dt \right| + \left| \int_0^T \langle B u_{lo}, \chi''_1 u_{lo} \rangle dt \right| \\ &\quad + \left| \int_0^T \langle (D_t + P_k - V) u_{lo}, (\chi_1 f)' u_{lo} \rangle dt \right|. \end{aligned}$$

We will examine each line of this estimate separately. The key thing to observe is that, since $B = f(x)\partial_x$, the first line in (4.4) has the highest number of derivatives so will require the most work. The terms with just ∂_x derivatives can be controlled by our initial estimate in Lemma 3.2. Hence, due to perfect local smoothing in the x direction and energy estimates, we can bound the two terms on the middle line of (4.4):

$$(4.5) \quad \left| \int_0^T \langle B u_{lo}, 2\chi'_1 \partial_x u_{lo} \rangle dt \right| + \left| \int_0^T \langle B u_{lo}, \chi''_1 u_{lo} \rangle dt \right| \leq C_T \|u_0\|_{H^{1/2}}^2.$$

Now for the first and last line in (4.4) we want to use the fact that

$$(D_t + P_k - V) u_{lo} = k L_1(\varphi u) - \frac{1}{k} L_2 \partial_\omega^2(\varphi u) - (1 - \psi)(2\varphi' \partial_x u + \varphi'' u).$$

We can use the fact that φ' and φ'' are compactly supported away from 0 and perfect local smoothing in the x direction to get that

$$(4.6) \quad \left| \int_0^T \langle 2(1 - \psi)\varphi' \partial_x u + (1 - \psi)\varphi'' u, (\chi_1 f)' u_{lo} \rangle dt \right| \leq C \|u_0\|_{H^{1/2}}^2$$

from the last line of (4.4), and

$$(4.7) \quad \left| \int_0^T \langle 2(1 - \psi)\varphi' \partial_x u + (1 - \psi)\varphi'' u, \chi B u_{lo} \rangle dt \right| \leq C \|u_0\|_{H^{1/2}}^2$$

from the first line of (4.4), for some constant C .

Next we want to handle the $k L_1(\varphi u) - \frac{1}{k} L_2 \partial_\omega^2(\varphi u)$ term coming from the last line in (4.4). To do this we can use the fact that χ_1 and χ'_1 are supported away from $x = 2$ so that we have perfect local smoothing in the ω direction according to Lemma 3.2. Hence

$$(4.8) \quad \left| \int_0^T \left\langle \frac{1}{k} L_2 \partial_\omega^2(\varphi u), (\chi_1 f)' u_{lo} \right\rangle dt \right| \leq C_T \frac{1}{|k|} \|u_0\|_{H^{1/2}}^2.$$

Now let $\tilde{\psi}$ be a smooth, even, compactly supported bump function with $\tilde{\psi}(s) \equiv 1$ on $\text{supp}(\psi(s))$. Let $\tilde{\chi}_1$ be a smooth compactly supported function such that $\tilde{\chi}_1(s) \equiv 1$ on the support of χ_1 but still supported away from $x = 0$ and $x = 2$. Then

$$\tilde{\psi}(D_x/k) L_1 \tilde{\psi}(D_x/k) = L_1 + \mathcal{O}(|k|^{-\infty}),$$

which gives

$$\begin{aligned}
& \left| \int_0^T \langle k L_1(\varphi u), (\chi_1 f)' u_{\text{lo}} \rangle dt \right| \\
&= \left| \int_0^T \left\langle \tilde{\chi}_1 k \tilde{\psi} L_1 \tilde{\psi}(\varphi u), (\chi_1 f)' u_{\text{lo}} \right\rangle dt \right| + C_T \|u_0\|_{H^{1/2}}^2 \\
&= \left| \int_0^T \left\langle \tilde{\chi}_1 k L_1 \tilde{\psi}(\varphi u), (\chi_1 f)' \tilde{\psi}(\varphi u) \right\rangle dt \right| + C_T \|u_0\|_{H^{1/2}}^2 \\
(4.9) \quad &\leq C \int_0^T |k| \|\tilde{\chi}_1 \tilde{\psi}(\varphi u)\|^2 dt + C_T \|u_0\|_{H^{1/2}}^2.
\end{aligned}$$

Combining (4.6) with (4.8) and (4.9), we estimate the the last line in (4.4):

$$\begin{aligned}
& \left| \int_0^T \langle (D_t + P_k - V) u_{\text{lo}}, (\chi_1 f)' u_{\text{lo}} \rangle dt \right| \\
(4.10) \quad &\leq C \int_0^T |k| \|\tilde{\chi}_1 \tilde{\psi}(\varphi u)\|^2 dt + C_T \|u_0\|_{H^{1/2}}^2.
\end{aligned}$$

We now proceed with the first line in (4.4). We have already estimated the lowest order parts in (4.7). We will deal with the term with L_1 last. That means we need to estimate

$$\left| \int_0^T \left\langle \chi_1 B u_{\text{lo}}, \frac{1}{k} L_2 \partial_\omega^2(\varphi u) \right\rangle dt \right|.$$

The difficulty is that there is one x derivative in B and two ω derivatives. We expect the $1/k$ to essentially remove one derivative to use Lemma 3.2 away from $x = 1$. However, this requires some careful observations.

Since the wavefront set of L_2 is contained where $\psi' \neq 0$, we have $|D_x/k| \sim \varepsilon > 0$ on the wavefront set of ψ' . Recall that $\tilde{\chi}_1$ is a bump function satisfying $\tilde{\chi}_1 \equiv 1$ on $\text{supp } \chi_1$ but $\tilde{\chi}_1 \equiv 0$ near $x = 2$. We also choose a bump function $\tilde{\psi}_1(r)$ satisfying $\tilde{\psi}_1(r) \equiv 1$ on $\text{supp } \psi'(r)$ but $\tilde{\psi}_1(r) \equiv 0$ near $r = 0$. The point is that then $(\frac{\partial_x}{k}) \tilde{\psi}_1(D_x/k)$ is a bounded operator on L^2 .

Then

$$\begin{aligned}
& \int \langle \chi_1 B u_{\text{lo}}, k^{-1} L_2 \partial_\omega^2(\varphi u) \rangle dt \\
&= \int \left\langle \chi_1 k^{-1} f(x) \partial_x \tilde{\psi}_1 \left(\frac{D_x}{k} \right) \partial_\omega u_{\text{lo}}, \tilde{\chi}_1 L_2 \partial_\omega u \right\rangle dt + O(k^{-\infty}) \int_0^T \|\tilde{\chi}_1 \partial_\omega u\|^2 dt \\
(4.11) \quad &\leq C \int \left(\left\| \chi_1(x) \left(\frac{\partial_x}{k} \right) \tilde{\psi}_1 \left(\frac{D_x}{k} \right) \partial_\omega u_{\text{lo}} \right\|^2 + \|\tilde{\chi}_1 L_2 \partial_\omega(\varphi u)\|^2 \right) dt + O(k^{-\infty}) \int_0^T \|\tilde{\chi}_1 \partial_\omega u\|^2 dt.
\end{aligned}$$

The operator

$$\chi_1(x) \left(\frac{\partial_x}{k} \right) \tilde{\psi}(D_x/k)$$

is bounded on L^2 and supported away from $x = 2$. Similarly, the operator $\tilde{\chi}_1 L_2$ is bounded on L^2 and supported away from $x = 2$. The $\mathcal{O}(|k|^{-\infty})$ term has $\tilde{\chi}_1 \partial_\omega u$, which is again supported away from $x = 2$. Lemma 3.2 guarantees perfect local smoothing in the ω direction away from $x = 2$, so applying Lemma 3.2 to (4.11) yields

$$(4.12) \quad \left| \int \langle \chi_1 B u_{\text{lo}}, k^{-1} L_2 \partial_\omega^2(\varphi u) \rangle dt \right| \leq C \|u_0\|_{H^{1/2}}^2.$$

Combining (4.3) with (4.5), (4.10), and (4.12), we have

$$\begin{aligned}
\int_0^T \langle \chi_1 k^2 u_{\text{lo}}, u_{\text{lo}} \rangle dt &\leq C_T \|u_0\|_{H^{1/2}(X)}^2 + C \left| \int_0^T \langle \chi_1 [D_t + P_k - V, B] u_{\text{lo}}, u_{\text{lo}} \rangle dt \right| \\
&\leq C_T \|u_0\|_{H^{1/2}}^2 + C \left| \int_0^T \langle \chi_1 B u_{\text{lo}}, k L_1(\varphi u) \rangle dt \right| \\
&\leq C_T \|u_0\|_{H^{1/2}}^2 + C \left| \int_0^T \langle \chi_1 B u_{\text{lo}}, \tilde{\chi}_1 \tilde{\psi} k L_1(\varphi u) \rangle dt \right| \\
&\leq C_T \|u_0\|_{H^{1/2}}^2 + C \int_0^T \|\tilde{\chi}_1 \tilde{\psi} k L_1(\varphi u)\|^2 dt \\
(4.13) \quad &\leq C_T \|u_0\|_{H^{1/2}}^2 + C \int_0^T \|\tilde{\chi}_1 \tilde{\psi} k(\varphi u)\|^2 dt.
\end{aligned}$$

Finally, we observe that, since $u_{\text{hi}} = \psi(\varphi u)$, we have

$$\|k \chi_1 u_{\text{hi}}\| \leq \|k \tilde{\chi}_1 u_{\text{hi}}\| = \|k \tilde{\chi}_1 \tilde{\psi}(\varphi u)\| + \mathcal{O}(1) \|u\|.$$

That means

$$(4.14) \quad \int_0^T \|k \chi_1 u_{\text{hi}}\|^2 dt \leq C \int_0^T \|k \tilde{\chi}_1 \tilde{\psi}(\varphi u)\|^2 dt + C_T \|u_0\|_{H^{1/2}}^2.$$

According to (4.13), we can estimate the low frequency part of u in terms of a quantity similar to the high frequency estimate (4.14). So for both u_{hi} and u_{lo} , it suffices to estimate the high frequency part:

$$\int_0^T \|k \tilde{\chi}_1 \tilde{\psi}(\varphi u)\|^2 dt$$

where $\tilde{\chi}_1$ is supported near $x = 1$ and $\tilde{\psi}$ has compact support.

5. THE HIGH FREQUENCY ESTIMATE

We use the FF^* type argument employed in [CW13] and [CM14]. Let us drop the tilde notation and consider functions $\chi(x)$ supported near $x = 1$ and supported away from $x = 0$ and $x = 2$, as well as $\psi(D_x/k)$ micro-supported near 0. Let $F(t)$ be defined by

$$F(t)g = \chi(x)\psi(D_x/k)k^r e^{-it(P_k - V)}g,$$

where $e^{-it(P_k - V)}$ is the free propagator. We want to show that for $r = \frac{2}{2m+3}$ we have a mapping $F : L_x^2 \rightarrow L^2([0, T])L_x^2$, since then

$$\|k^{1-r} F(t)u_0\|_{L^2([0, T])L_x^2} \leq C \|k^{1-r} u_0\|_{L^2}$$

is the desired local smoothing estimate. We have such a mapping if and only if $FF^* : L^2([0, T])L_x^2 \rightarrow L^2([0, T])L_x^2$. We compute

$$FF^*f(x, t) = \psi(D_x/k)\chi(x)k^{2r} \int_0^T e^{i(t-s)(P_k - V)} \chi(x)\psi(D_x/k)f(x, s)ds,$$

and need to show that $\|FF^*f\|_{L^2 L^2} \leq C \|f\|_{L^2 L^2}$. Now write $FF^*f(x, t) = \psi\chi(v_1 + v_2)$, where

$$v_1 = k^{2r} \int_0^t e^{i(t-s)(P_k - V)} \chi(x)\psi(D_x/k)f(x, s)ds,$$

and

$$v_2 = k^{2r} \int_t^T e^{i(t-s)(P_k - V)} \chi(x)\psi(D_x/k)f(x, s)ds,$$

so that

$$(D_t + P_k - V)v_j = \pm ik^{2r}\chi\psi f,$$

and it suffices to estimate

$$\|\psi\chi v_j\|_{L^2 L^2} \leq C\|f\|_{L^2 L^2}.$$

Now taking the Fourier transform in time and using Plancheral's theorem, we have that it suffices to estimate

$$\|\psi\chi\hat{v}_j\|_{L^2 L^2} \leq C\|\hat{f}\|_{L^2 L^2}$$

but this is the same as estimating

$$\|\psi\chi k^{2r}(\tau \pm i0 + P_k - V)^{-1}\chi\psi\|_{L_x^2 \rightarrow L_x^2} \leq C.$$

This means that for the operator P_k defined above we can reduce the estimate to showing that

$$\|\psi\chi k^{2r}(\tau \pm i0 + P_k - V)^{-1}\chi\psi\|_{L_x^2 \rightarrow L_x^2} \leq C.$$

Let $-z = \tau k^{-2}$ and $h = k^{-1}$ to get

$$\|\psi\chi(-z \pm i0 + (hD_x)^2 + V_1 - h^2V_2\partial_\omega + h^2V)^{-1}\chi\psi\|_{L_x^2 \rightarrow L_x^2} \leq C.$$

In particular we want to show that

$$(5.1) \quad \|(-z + (hD_x)^2 + V_1 - h^2V_2\partial_\omega + h^2V)\chi(x)\psi(hD_x)\varphi u\|_{L^2}^2 \geq h^{\frac{4m+2}{2m+3}}\|\chi(x)\psi(hD_x)(\varphi u)\|_{L^2}^2.$$

So with the following lemma we can get the desired result.

Lemma 5.2. *For $\varepsilon > 0$ sufficiently small, let $\varphi \in S(T^*\mathbb{R})$ have compact support in $\{|(x-1, \xi)| \leq \varepsilon\}$. Then there exists $C_\varepsilon > 0$ such that*

$$(5.3) \quad \|(P - z)\varphi^w u\|^2 \geq C_\varepsilon h^{(4m+2)/(2m+3)}\|\varphi^w u\|^2, \quad z \in [C - \varepsilon, C + \varepsilon]$$

where $P = (hD_x)^2 + V_1 - h^2V_2\partial_\omega^2 - h^2V$ and φ^w denotes quantization in only the x and ∂_x directions and $\|\cdot\|$ denotes the L^2 norm in x and ω coordinates.

Now when looking at the norm we absorb the h^2V term to the right hand side of (5.3) since $(4m+2)/(2m+3) < 2$. We just need to deal with the $-h^2V_2\partial_\omega^2$ term because

$$\|((hD_x)^2 + V_1)\varphi^w u\|_{L^2}^2 \geq h^{(4m+2)/(2m+3)}\|\varphi^w u\|_{L^2}^2$$

by [CM14].

Now we very briefly summarize the commutator process as in [CM14]. We define Λ, Λ_2 as follows: freeze $\epsilon_0 > 0$ and let

$$\Lambda(r) = \int_0^r \langle t \rangle^{-1-\epsilon_0} dt, \quad \Lambda_2(r) = \int_{-\infty}^r \langle t \rangle^{-1-\epsilon_0} dt.$$

For the remainder of the paper, we denote by $\chi(s)$ a smooth, even bump function with $\chi(s) \equiv 1$ for $|s| \leq \delta_1$ and support in $\{|s| \leq 2\delta_1\}$. Here $\delta_1 \gg \epsilon$ where $\epsilon > 0$ is as in Lemma 5.2. with compact support near 0 so that $\chi(x-1)\chi(\xi)$ microlocalizes (in the semi-classical sense) near $x = 1$ and $\xi = 0$. Just as in [CM14], let $\tilde{h} \gg h$ be a second small parameter and let

$$a(x, \xi; h, \tilde{h}) = \Lambda(\Xi)\Lambda_2(X-1)\chi(x-1)\chi(\xi)$$

where

$$X-1 = \frac{x-1}{(h/\tilde{h})^\alpha}, \quad \Xi = \frac{\xi}{(h/\tilde{h})^\beta}.$$

Here

$$\alpha = \frac{2}{2m_1 + 3}, \quad \beta = 1 - \alpha.$$

We now employ a similar commutator method to get a favorable sign on the V_2 term. To somewhat ease notation, let $v = \varphi^w u$. We have that

$$(5.4) \quad C\|v\|\|Pv\| \geq \langle [P, a^w], v, v \rangle = \langle i[(hD_x)^2 + V_1, a^w]v, v \rangle + \langle i[-h^2 V_2, a^w]\partial_\omega^2 v, v \rangle$$

for some constant C . The first term is exactly the same as in [CM14]. We have assumed that V_2 is decreasing and linear near $x = 1$, so if $\epsilon > 0$ is sufficiently small, V_2'' is supported away from the support of $\chi(x - 1)$. In particular, we have $i[-V_2, a^w] = -h(H_{V_2}a)^w$, where

$$-H_{V_2}a = V_2'(x)\partial_\xi a \leq 0$$

on the wavefront set of v . Hence

$$\langle (-H_{V_2}a)^w \partial_\omega^2 v, v \rangle = \langle (H_{V_2}a)^w \partial_\omega v, \partial_\omega v \rangle \geq 0.$$

Plugging in to (5.4), this implies that

$$C\|v\|\|Pv\| \geq \langle i[P, a^w]v, v \rangle \geq \langle i[hD_x^2 + V_1, a^w]v, v \rangle.$$

Hence, by the results in [CM14] we have that

$$\|v\|\|Pv\| \geq \frac{1}{C}h^{(4m+2)/(2m+3)}\|v\|^2 + \frac{C'}{C}h^{3-\beta}\|\partial_\omega v\|^2$$

This completes the proof when separating variables in the θ direction. Separating variables in just the ω direction is similar.

REFERENCES

- [Bur04] N. Burq, *Smoothing effect for Schrödinger boundary value problems*, Duke Math. J. **123** (2004), no. 2, 403–427. MR2066943 (2006e:35026) ↑4
- [Chr07] Hans Christianson, *Semiclassical non-concentration near hyperbolic orbits*, J. Funct. Anal. **246** (2007), no. 2, 145–195. MR2321040 ↑4
- [Chr08] ———, *Dispersive estimates for manifolds with one trapped orbit*, Comm. Partial Differential Equations **33** (2008), 1147–1174. ↑4
- [Chr11] ———, *Quantum monodromy and nonconcentration near a closed semi-hyperbolic orbit*, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3373–3438. MR2775812 (2012b:58051) ↑4
- [Chr18] ———, *High-frequency resolvent estimates on asymptotically Euclidean warped products*, Comm. Partial Differential Equations **43** (2018), no. 9, 1617–1645. ↑4, 5
- [CKS95] Walter Craig, Thomas Kappeler, and Walter Strauss, *Microlocal dispersive smoothing for the Schrödinger equation*, Comm. Pure Appl. Math. **48** (1995), no. 8, 769–860. ↑4
- [CM14] Hans Christianson and Jason Metcalfe, *Sharp local smoothing for warped product manifolds with smooth inflection transmission*, Indiana Univ. Math. J. **63** (2014), no. 4, 969–992. MR3263918 ↑1, 2, 4, 5, 12, 13, 14
- [CS88] Peter Constantin and Jean-Claude Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc. **1** (1988), no. 2, 413–439. ↑4
- [CW13] Hans Christianson and Jared Wunsch, *Local smoothing for the Schrödinger equation with a prescribed loss*, Amer. J. Math. **135** (2013), no. 6, 1601–1632. MR3145005 ↑4, 5, 6, 12
- [Dat09] Kiril Datchev, *Local smoothing for scattering manifolds with hyperbolic trapped sets*, Comm. Math. Phys. **286** (2009), no. 3, 837–850. MR2472019 (2009j:58042) ↑4
- [Doi96] Shin-ichi Doi, *Remarks on the Cauchy problem for Schrödinger-type equations*, Comm. Partial Differential Equations **21** (1996), no. 1–2, 163–178. MR1373768 (96m:35058) ↑4
- [iD96] Shin-ichi Doi, *Smoothing effects of Schrödinger evolution groups on riemannian manifolds*, Duke Math. J. **82** (1996), no. 3, 679–706. ↑4
- [Sjö87] Per Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. **55** (1987), no. 3, 699–715. ↑4
- [Veg98] Luis Vega, *Schrödinger equations: pointwise convergence to the initial data*, Proc. Amer. Math. Soc. **102** (1998), no. 4, 874–878. ↑4