Vector Fields (17.1)

July 12, 2020

Big Picture for today

Big Picture: A vector field assigns a *vector* at each point in a domain. Equivalently, it is a vector valued function of more than one variable.

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Write this as a *function* of (x, y):

$$\mathbf{F}(x, y) = \langle x, y\rangle\,,\;\; \mathbf{G}(x, y) = \langle -y, x\rangle\,.$$

Vector fields in $\mathbb{R}^2$

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$\mathbf{F}$ is continuous if f and g are continuous. $\mathbf{F}$ is differentiable if f and g are differentiable.

Example

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Example: $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ is a radial vector field in $\mathbb{R}^3$.

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Examples: If $\phi(x, y) = x + 2xy$, then $\nabla\phi(x, y) = \langle 1 + 2y, 2x\rangle$.

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At each point (x_0, y_0) in a circle, $\nabla\phi(x_0, y_0)$ is perpendicular to the level set $\phi(x, y) = x_0^2 + y_0^2$. Perpendicular to a circle means pointing outward.
$\rightsquigarrow$ $\mathbf{F}(x, y) = \nabla\phi(x, y) = \langle 2x, 2y\rangle$ is radial!

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