# LOCAL SMOOTHING OF THE SCHRÖDINGER EQUATION ON A MULTI-WARPED PRODUCT WITH DEGENERATE TRAPPING

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## ABSTRACT

Derrick Nowak: Local Smoothing of the Schrödinger Equation on a Multi-Warped Product with Degenerate Trapping
(Under the direction of Hans Christianson)

Geodesic trapping is an obstruction to dispersive estimates for solutions to the Schrödinger equation. In [CW13], Christianson and Wunsch prove a local smoothing estimate on a surface of revolution with degenerate trapping. In this thesis, we look to extend this result to the case of a multi-warped product with two infinite directions. A multi-warped product manifold with one infinite direction was used in [CN22], where a local smoothing result was proven for inflection-transmission type trapping studied initially by Christianson and Metcalfe in [CM14]. We construct a multi-warped product with two infinite ends where each warped piece has degenerate trapping at different points in the radial direction. In the inflection-transmission type trapping case, the trapping in each warped direction did not interact leaving the trapping at just two points in the radial direction. However, in this thesis, the trapping is complicated. The projection of all trapped trajectories onto the radial direction after separating variables will be a countable dense subset of points in the interval $[-\varepsilon, \varepsilon]$. The main result of this thesis is to show that while the trapping is more complicated, we gain the same local smoothing estimates from [CW13] in each angular direction.

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# CHAPTER 1

# INTRODUCTION

A solution $u$ of the Schrödinger equation on $\mathbb{R}^n$ with initial condition $u_0 \in \mathscr{S}(\mathbb{R}^n)$ is a solution to the equations

$$\begin{cases} (D_t - \Delta)u(t,x) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

where $D_t = \frac{1}{i}\partial_t$ and $\mathscr{S}$ is that set of Schwartz functions defined in Definition 2.3.1. The Schrödinger equation describes the wave formulation of quantum particles. In the form above $u_0(x)$ would be an initial probability that a given particle appears at $x$ and $\Delta u(t,x)$ is the kinetic energy of the particle at $(t,x)$. The equation tells us that the evolution of $u(t,x)$ is related to the kinetic energy and is the quantum analog for Newton's second law. In $\mathbb{R}^n$, a solution to the Schrödinger equation is given by

$$\begin{aligned} u(t,x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}e^{-it\xi^2}\int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle}u_0(y)dyd\xi \\ &= e^{it\Delta}u_0(x). \end{aligned}$$

This gives an exact solution for $u_0 \in \mathscr{S}$ to the equation and allows us to model free particles. However, there is often a potential energy term $V(x)$. In this situation the Schrödinger equation becomes

$$\begin{cases} (D_t - \Delta + V)u(t,x) = 0 \\ u(0,x) = u_0(x). \end{cases}$$

In many cases of Schrödinger equations with a potential there is no known explicit solution, so estimates are used to understand how solutions behave. Instead of having a potential term, another way the Schödinger equation might be altered is by changing the underlying geometry. We might want to study particles on a Riemannian manifold $M$ without boundary with metric $g$. In this situation the Schrödinger equation becomes

$$\begin{cases} (D_t - \Delta_g)u(t,x) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

where $\Delta_g$ is the Laplacian on $M$ with metric $g$. The change in the geometry is similar to adding a potential term. In this thesis, we will be able to reduce studying the Schrödinger equation on a 3-dimensional multi-warped product into studying the Schrödinger equation on $\mathbb{R}$ with a potential term.

The Schrödinger equation is a type of dispersive equation, which are equations with solutions that propagate based on the frequency of oscillation. Specifically in the Schrödinger equation case, waves propagate proportionally to the frequency of oscillation. Another common dispersive equation is the wave equation, which propagates in the direction of the frequency of oscillation, but has constant speed of propagation. In contrast, the heat equation and other parabolic and elliptic equations are not dispersive equations and do not propagate based on the frequency of oscillation. A main feature of dispersive equations is that the way waves propagate governs how solutions evolve over time. On a Riemannian manifold, the waves for dispersive equations propagate in the direction of geodesics, so the nature of the geodesics is important to determining the behavior of dispersive equations. On $\mathbb{R}^n$, since the geodesics are straight lines, we should expect solutions to the Schrödinger equation to spread out over time and go off towards infinity. However, if there are trapped geodesics that do not go off towards infinity, then we should expect the solution to spread out slower over time. This difference in the behavior of geodesics affects the regularity of solutions to the Schrödinger equation locally in space and averaged over time. For example, a dispersive equation with an initial condition concentrated near a trapped geodesic will disperse slowly near this trapped geodesic and affect how quickly the solution will spread out towards infinity. Since the Schrödinger equation conserves energy, we cannot have any global gain in regularity in $x$ however, we can analyze how regularity is affected locally in $x$ over time. This type of estimate is what we consider a local smoothing estimate. Specifically, if $u$ is a solution to the Schrödinger equation on $\mathbb{R}^n$ with initial condition $u_0 \in \mathscr{S}(\mathbb{R}^n)$, then the local smoothing estimate is of the form

$$\int_0^T \sum_{i=1}^n ||\langle x_i \rangle^{-1} \partial_i u||^2_{L^2_x(\mathbb{R}^n)} \leq C_T ||u_0||^2_{H^{1/2}(\mathbb{R}^n)}.$$

where $C_T > 0$ is a constant dependent on $T$ and $\langle x_i \rangle^{-1} = (1 + x_i^2)^{-1/2}$. This type of estimate shows that over time locally $u(t,x)$ gains $1/2$ derivatives. The term $\langle x_i \rangle^{-1}$ localizes in space, because $\langle x_i \rangle^{-1} \to 0$ as $|x_i| \to \infty$. The term $\langle x_i \rangle^{-1}$ is not optimal, but agrees with what is shown in Theorem 3.2.1.

We will briefly provide a heuristic estimate for the existence of local smoothing estimates for dispersive equations on $\mathbb{R}$. A formal proof can be found in Section A.4. Consider the differential equation $(D_t + D_x^m)u(t,x) = 0$ for $m \in \mathbb{N}$ with initial condition $u(0,x) = u_0(x) \in \mathscr{S}(\mathbb{R})$ where $D_t = \frac{1}{i}\partial_t$ and $D_x^m = \frac{1}{i^m}\partial_x^m$. Then, a solution to the equation will have the form

$$u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x-y)\xi} e^{-it\xi^m} u_0(y) dy d\xi.$$

Combining exponentials we see that

$$u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\varphi(t,x,y,\xi)} u_0(y) dy d\xi$$

for the phase function

$$\varphi = (x-y)\xi - t\xi^m.$$

Rapid decay of oscillatory integrals says that the solutions will be concentrated where

$\partial_\xi \varphi = 0$. A brief review of rapid decay of oscillatory integrals and stationary phase can be found in Theorem A.2.1 and Theorem A.2.2 in the Appendix. More general results on stationary phase can be found in [Zwo12]. Suppose that we have some initial condition $u_0$ such that $\hat{u}_0(\xi)$ is concentrated near $\xi_0 > 0$ and $u_0(x)$ is concentrated near $y_0$ in some interval $I$ of length $c$. Since $\partial_\xi \varphi = (x-y) - mt\xi^{m-1}$, we have that $\partial_\xi \varphi(t, x, y_0, \xi_0) = 0$ when $x = y_0 + mt\xi_0^{m-1}$. This implies that $u(t,x)$ is concentrated near $x = y_0 + mt\xi_0^{m-1}$. Hence $u$ propagates at speed $m\xi_0^{m-1}$. When $mt\xi_0^{m-1} > c$, $u(t,x)$ will be concentrated away from the interval $I$. This means when $t > \frac{c}{m}\xi_0^{-(m-1)}$, $u(t,x)$ will be small on $I$. Then roughly we will get,

$$\int_0^T \int_I |\langle D_x \rangle^{(m-1)/2} u|^2 dx dt = \int_0^{\frac{c}{m}\xi_0^{-(m-1)}} \int_I |\langle D_x \rangle^{(m-1)/2} u|^2 dx dt + \text{small}.$$

Using Plancherel's theorem,

$$\begin{aligned} \int_0^{\frac{c}{m}\xi_0^{-(m-1)}} \int_I |\langle D_x \rangle^{(m-1)/2} u|^2 dx dt &\leq \int_0^{\frac{c}{m}\xi_0^{-(m-1)}} \int_{\mathbb{R}} |\langle D_x \rangle^{(m-1)/2} u|^2 dx dt \\ &= C \int_0^{\frac{c}{m}\xi_0^{-(m-1)}} \int_{\mathbb{R}} |\langle \xi \rangle^{(m-1)/2} \hat{u}(t,\xi)|^2 d\xi dt. \end{aligned}$$

Now, $\hat{u}(t,\xi)$ is concentrated near $\xi_0$, so $|\xi^{(m-1)/2}\hat{u}(t,x)|$ is roughly $|\xi_0^{(m-1)/2}\hat{u}(t,x)|$. Thus,

$$\begin{aligned} \int_0^{\frac{c}{m}\xi_0^{-(m-1)}} \int_{\mathbb{R}} |\langle \xi \rangle^{(m-1)/2} \hat{u}(t,\xi)|^2 d\xi dt &\leq C \int_0^{\frac{c}{m}\xi_0^{-(m-1)}} \int_{\mathbb{R}} |\xi_0^{(m-1)/2} \hat{u}(t,\xi)|^2 d\xi dt \\ &\leq \xi_0^{-(m-1)} \xi_0^{(m-1)} C \sup_{t \in [0,T]} \{ \int_{\mathbb{R}} |\hat{u}(t,\xi)|^2 d\xi \} \\ &= C \sup_{t \in [0,T]} \{ ||u(t,\cdot)||_{L^2}^2 \}. \end{aligned}$$

Due to the conservation of mass of solutions to dispersive equation,

$$||u(t,\cdot)||_{L^2}^2 = ||u_0||_{L^2}^2$$

for all $t \in \mathbb{R}$. Combining the estimates gives

$$\int_0^T \int_I |\langle D_x \rangle^{(m-1)/2} u|^2 dx dt \leq C||u_0||_{L^2}^2.$$

This heuristic shows that we should expect a local smoothing effect in $\mathbb{R}$. Specifically, in the Schrödinger equation situation $m = 2$, we should expect a gain of a $1/2$ derivative. The $m = 1$ case is the half wave operator, which has no local smoothing effect, and the $m = 3/2$ case is the water wave equation with surface tension, which gains $1/4$ derivatives.

### 1.1 Brief History of Local Smoothing

Local smoothing estimates for the Schrödinger equation have been studied in many different contexts. Local smoothing was first observed by Kato [Kat83] for the KdV equation. Local smoothing of the Schrödinger equation and more general dispersive equations were later studied by Constantin-Saut [CS88], Sjölin [Sjö87], Vega [Veg88], Kato-Yajima [KY89], and Jourńe-Soffer-Sogge [JSS91]. On $\mathbb{R}^n$ there exists a $C_T > 0$ such that

$$\int_0^T \sum_{i=1}^n ||\langle x_i \rangle^{-1} \partial_i u||_{L_x^2(\mathbb{R}^n)}^2 dt \leq C_T ||u_0||_{H^{1/2}(\mathbb{R}^n)}^2.$$

This result was extended to asymptotically Euclidean non-compact manifolds where the geodesic flow is non-trapping in [Doi96], [CKS95]. Doi also showed there must be a loss in regularity if there is trapping. These results show that a gain of $1/2$ derivatives in the local smoothing estimate is the maximum and any trapped geodesic implies that the gain must be less than $1/2$. Local-in-space smoothing estimates for the Schrödinger equation on smooth manifolds with asymptotic flatness conditions were extended to global-in-time estimates in [RT07] and [MMT08]. Work by Ralston [Ral71] implied that if the manifold has stable trapping, then there can be no polynomial gain in derivatives. In contrast, work in [Bur04], [Chr07], [Chr08], and [Dat09] showed that if the manifold has non-degenerate unstable trapping, then there is a $1/2 - \varepsilon$ gain in derivatives. There are many local smoothing results, however, we are going to focus on extending the estimates on manifolds with degenerate unstable trapping in [CW13] and [CM14]. In [CW13], the authors considered a surface of revolution $M = \mathbb{R} \times \mathbb{S}^1$ with metric

$$g = dx_2^2 + A(x)^2 d\theta^2$$

where $A(x)^2$ has a local minimum of order $2m$, where $m$ is an integer greater than 1. We say that there is unstable degenerate trapping at the origin of order $2m$. In this case, we gain only $1/(m+1)$ instead of $1/2 - \varepsilon$ derivatives. In [CM14], the authors extend the result from [CW13] to the case of a surface of revolution where $A(x)^2$ has a local minimum of order $2m_1$ at $x = 0$ and an inflection point of order $2m_2 + 1$ at $x = 1$, where $m_1$ and $m_2$ are integers greater than 1. There is degenerate trapping of order $2m_1$ at $x = 0$ and we say that there is inflection-transmission type trapping of order $2m_2 + 1$ at $x = 1$. They were able to show that the gain in derivatives is the minimum of $1/(m_1 + 1)$ and $2/(2m_2 + 3)$.

In [CN22], the author with Christianson followed a similar argument to [CM14] to show local smoothing results for the multi-warped product manifold $X = \mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{S}^1$ with metric

$$g(x, \theta, \omega) = dx^2 + A_1(x)^2 d\theta^2 + A_2(x)^2 d\omega^2$$

where there is inflection-transmission trapping at $x = 1$ in the $\theta$ direction and at $x = 2$ in the $\omega$ direction. In this case, we saw that the trapping was isolated in each direction. If a geodesic had non-zero initial velocity in both the $\theta$ and $\omega$ directions, then the geodesic was not trapped. If there is inflection-transmission trapping of order $2m_1 + 1$ at $x = 1$ in the $\theta$ direction and inflection-transmission type trapping of order $2m_2 + 1$ at $x = 2$ in the $\omega$ direction, then there is a $1/2$ gain in derivative away from the trapping. Overall we gain $2/(2m_1 + 3)$ derivatives in the $\theta$ direction and we gain $2/(2m_2 + 3)$ derivatives in the $\omega$ direction. The trapping was only at $x = 1$ and $x = 2$ allowing a positive commutator argument to isolate to the trapping.

In this thesis, we are going to prove local smoothing estimates for a multi-warped product with two infinite directions. In this situation the projection of all trapped trajectories onto the $x$-direction after separating variables is a countable dense subset of points in the interval $[-\varepsilon, \varepsilon]$. However, due to the extra dimension, we will show that we gain the same result from [CW13] in each angular direction. Let $M = \mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$ with metric

$$g(x, \theta, \omega) = dx^2 + U_+(x)^2 d\theta^2 + U_-(x)^2 d\omega^2$$

where $U_\pm$ are defined in Section 4. The main properties are that $U_\pm^{-2}$ have degenerate unstable critical points of order 4 at $\pm\varepsilon$ respectively and that $g(x, \theta, \omega)$ is Euclidean for $|x| \geq 4\varepsilon$. The main result of the thesis is the following,

**Theorem 1.1.1.** *Let $M$ be the multi-warped product with $\Delta_M$ constructed in Section 4. Let $u$ be a solution to the Schrödinger equation on $M$ with initial condition $u_0 \in \mathscr{S}(M)$. For each $T > 0$ there exists a constant $C$ such that*

$$\int_0^T ||\langle x \rangle^{-3/2} u||^2_{H^1(M)} dt \leq C ||u_0||^2_{H^{2/3}(M)}.$$

We begin by providing background on multi-warped products, the Schrödinger equation, and pseudo-differential operators. We will define multi-warped products in Section 2.1 and explain why they are a useful structure to study. Next, we will develop the necessary pseudo-differential calculus in Section 2.3, which will include the two parameter calculus introduced in [CW13]. Finally, we will explain the Schrödinger equation and why we study local smoothing in Section 2.5.

We will then prove the result of local smoothing on $\mathbb{R}^2$ to show the positive commutator technique, which will play a role in the results discussed later in Section 3.3. We will then go over the result in [CW13] in Section 3.3. This will include a discussion of the proof strategy.

Afterwards, we will discuss the result from [CM14] in Section 3.4.1. We will discuss the first results on multi-warped products in [CN22] and explain why this result is more straightforward than the result in this thesis in Section 3.4.2. Finally, we will build up from the previous results to prove our local smoothing result on a multi-warped product where the projection of trapped trajectories onto the $x$-direction after separating variables is a countable dense subset of an interval rather than isolated to just two points.

# CHAPTER 2

# BACKGROUND

## 2.1 Multi-Warped Products

One of the simplest examples of a warped product manifold is a surface of revolution, which is given by revolving a curve. In this situation we have a manifold $M = I \times \mathbb{S}^1$ with a metric $g = dx^2 + f(x)g_{\mathbb{S}^1}$ for a function $f(x)$ and interval $I$. At each point the metric on $\mathbb{S}^1$ is warped by a function dependent only on $x$. The other classic example of a warped product manifold is $\mathbb{R}^n$ in polar coordinates. In polar coordinates $\mathbb{R}^n = \mathbb{R}_+ \times \mathbb{S}^{n-1}$ together with the metric

$$g = dx^2 + x^2 g_{\mathbb{S}^{n-1}}.$$

In this case the $x^2$ term is considered to be the warping function for the metric on $\mathbb{S}^{n-1}$, even if the end result is Euclidean space.

The advantage of warped product manifolds when the warped manifolds are compact is that we can separate variables. This is what leads us to introduce multi-warped products.

**Definition 2.1.1.** *Let $M_1, M_2, \ldots M_N$ be compact Riemannian manifolds without boundary. Denote the corresponding metrics $g_{M_1}, \ldots, g_{M_N}$ and suppose they have dimensions $n_1, \ldots, n_N$. Let $I$ be an interval on $\mathbb{R}$. Let $A_1, \ldots, A_N : I \to \mathbb{R}$ satisfying $A_j(x) > 0$ for all $j = 1, \cdots, N$. Let*

$$X = I \times M_1 \times M_2 \times \cdots \times M_N$$

*with the metric*

$$g = dx^2 + A_1(x)^2 g_{M_1} + \ldots + A_N(x)^2 g_{M_N}.$$

*Then $X$ is called a multi-warped product manifold with cross sections $M_1, \dots M_N$.*

**Remark 2.1.2.** *We will call the $x$ coordinate the radial direction. When $M_1, \cdots, M_n$ are $\mathbb{S}^1$, we will call those the angular directions.*

**Definition 2.1.3** (Multi-Warped Product One Infinite Direction)**.** *Let $M_1, M_2, \dots M_N$ be compact Riemannian manifolds without boundary. Denote the corresponding metrics $g_{M_1}, \dots, g_{M_N}$ and suppose they have dimensions $n_1, \dots, n_N$ respectively. Let $A_1, \dots, A_N : \mathbb{R}_+ \to \mathbb{R}$ satisfying $A_j(x) > 0$ for $j = 1, \cdots, N$. Let*

$$X = \mathbb{R}_+ \times M_1 \times M_2 \times \cdots \times M_N$$

*with the metric*

$$g = dx^2 + A_1(x)^2 g_{M_1} + \dots + A_N(x)^2 g_{M_N}.$$

*Then, $X$ is called a multi-warped product manifold with one infinite direction. We will call $X$ Euclidean outside of a compact set and near zero if $A_j(x) = x$ for $x \notin [a, b]$ for some positive integers $b > a > 0$ for $j = 1, \cdots, N$.*

This is an extension of the polar coordinates situation in the sense that $I = \mathbb{R}_+$. This type of manifold is used in [CN22]. We can also extend the situation where we have $I = \mathbb{R}$, so that the manifold has two infinite directions. The main result of this thesis is proving local smoothing estimates on a multi-warped product with two infinite directions.

**Definition 2.1.4** (Multi-Warped Product with Two Infinite Directions)**.** *Let $M_1, M_2, \dots M_N$ be compact Riemannian manifolds without boundary. Denote the corresponding metrics $g_{M_1}, \dots, g_{M_N}$ and suppose they have dimensions $n_1, \dots, n_N$ respectively. Let $A_1, \dots, A_N : \mathbb{R} \to \mathbb{R}$ satisfy $A_j(x) > 0$ for $j = 1, \cdots, N$. Let*

$$X = \mathbb{R} \times M_1 \times M_2 \times \cdots \times M_N$$

*with the metric*

$$g = dx^2 + A_1(x)^2 g_{M_1} + \ldots + A_N(x)^2 g_{M_N}.$$

*Then, $X$ is a multi-warped product manifold with two infinite directions. We will call $X$ Euclidean outside of a compact set if $A_j(x) = |x|$ for $|x| > C$ for some positive number $C$ for $j = 1, \cdots, N$.*

The multi-warped product in the thesis will be of the form $M = \mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$ with a metric

$$g = dx^2 + A_1^2(x) d\theta_1^2 + A_2^2(x) d\theta_2^2$$

where $A_1(x), A_2(x) > 0$ and $|A_1|, |A_2| = |x|$ when $|x| > C$ for some constant $C$.

## 2.2 Geodesics and Trapping

Let $M$ be a Riemannian manifold of dimension $n$ with metric $g$ and let $\gamma : I \to M$ be a curve where $\gamma(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ . Let $g^{ij}$ denote the $i, j$-th entry of the dual of the metric $g$. Then $\gamma(t)$ is a geodesic if and only if

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{dx_i}{dt} \frac{dx_j}{dt} = 0,\ k = 1, \cdots, n$$

where

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left\{ \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right\} g^{km}.$$

We will start by considering geodesics on a surface of revolution, since the geodesics can easily be visualized. We will consider the manifold $M = \mathbb{R} \times \mathbb{S}^1$ with metric $g = dx^2 + A_1(x)^2 d\theta_1^2$ where $A_1(x) > 0$. A trapped geodesic will be a geodesic of the form $\gamma(t) = (x(t), \theta_1(t))$ where there is a $C > 0$ such that $|x(t)| < C$ for all $t$.

**Definition 2.2.1.** *A trapped geodesic $\gamma(t)$ with initial conditions $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$ is stable if there exists an $\varepsilon > 0$ such that if $|(x, \xi) - (\tilde{x}, \tilde{\xi})| < \varepsilon$ then the geodesic $\tilde{\gamma}$ with initial* *conditions* $\tilde{\gamma}(0) = \tilde{x}$ *and* $\dot{\tilde{\gamma}}(0) = \tilde{\xi}$ *is trapped.*

For $M$ we will have

$$\Gamma^x_{\theta_1\theta_1} = -A_1 A_1',\ \Gamma^x_{ij} = 0 \text{ for all other } i, j.$$

The geodesic equation that governs $x(t)$ is given by

$$\frac{d^2x(t)}{dt^2} - A_1 A_1' \left(\frac{d\theta_1}{dt}\right)^2 = 0. \tag{2.2.2}$$

Notice from this equation that all periodic geodesics such that $\dot{x}(t) = 0$ for all $t$ are at critical points of $A_1$. In the case of [CW13] and [CM14] these periodic geodesics are the only trapped geodesics, however trapping can be much more complicated. A few examples of trapped geodesics are shown in Figure 2.1. We can have periodic geodesics without a fixed $x$ value. For example, on a sphere all great circles are periodic geodesics. This type of trapping is shown as the stable trapping in Figure 2.1. Stable trapping does not have to be periodic. The defining characteristic is that a small change in initial conditions for the geodesic will still result in a trapped geodesic. It is also possible to have non-periodic unstable trapped geodesics.

We covered the geodesic equations in terms of the metric, however geodesics can also be defined in terms of the dual of the metric. Let $M$ be a Riemannian manifold with local coordinates $x_1, \cdots x_n$, metric $g_{ij}$ and inverse metric $g^{ij}$. Note that we can view a point $\xi_1 dx_1 + \cdots + \xi_n dx_n$ in $T_x^*M$ as $(x, \xi) = (x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$. Given $(x, \xi)$ we define the Hamiltonian $H(x, \xi) = \sum_{a,b=1}^n \frac{1}{2} g^{ab}(x)\xi_a\xi_b$. Then, the Hamiltonian flow is given by

$$\begin{aligned}
\dot{x}_a &= \frac{\partial H}{\partial \xi_a} = \sum_{b=1}^n g^{ab}(x)\xi_b \\
\dot{\xi}_a &= -\frac{\partial H}{\partial x_a} = -\sum_{b,c=1}^n \frac{1}{2}\frac{\partial g^{bc}}{\partial x_a}\xi_b\xi_c
\end{aligned}$$

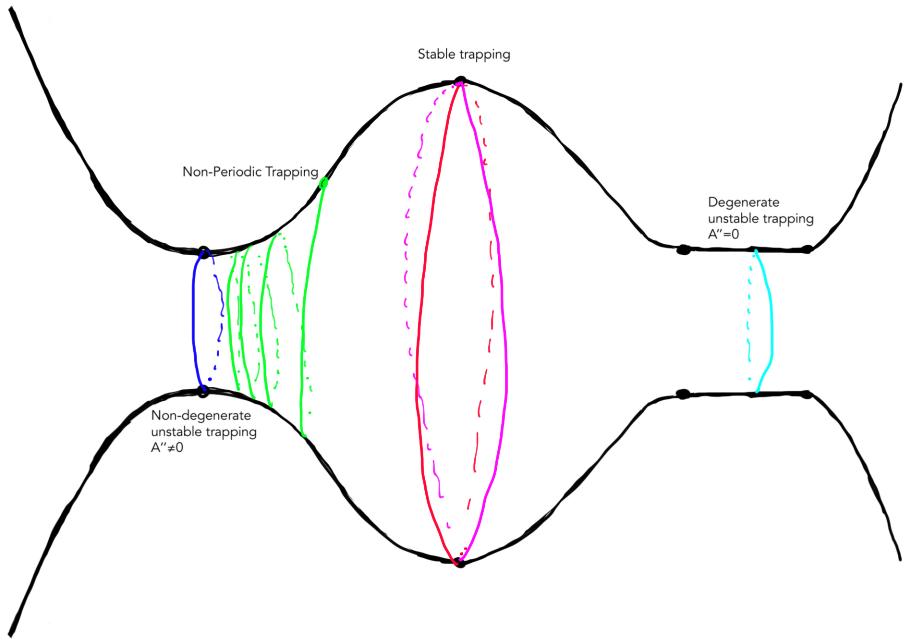


Figure 2.1: Examples of Trapped Geodesics

where $\dot{x}_a = \frac{d}{dt}x_a$. In the case of the surface of revolution above, the Hamiltonian flow is given by the following equations,

$$\begin{aligned}
\dot{x} &= \xi_x \\
\dot{\theta}_1 &= A_1^{-2}\xi_{\theta_1} \\
\dot{\xi}_x &= -\frac{1}{2}\partial_x(A_1^{-2})\xi_{\theta_1}^2 \\
\dot{\xi}_{\theta_1} &= 0.
\end{aligned}$$

Using $\xi_x(t) = \dot{x}(t)$ and $\xi_{\theta_1}(t) = A_1^2\dot{\theta}_1$ gives

$$\frac{d^2x}{dt^2} = A_1'A_1\dot{\theta}_1^2$$

which agrees with (2.2.2). The advantage of looking at geodesics in terms of Hamiltonian flow is that flows are deterministic. Additionally, we know that the Hamiltonian is preserved by the flow. This tells us that geodesics live on projections of level sets of the Hamiltonian. Note that $\dot{\xi}_{\theta_1} = 0$ for geodesics on a surface of revolution. We can take $\xi_{\theta_1} = 1$, since if $\xi_{\theta_1} = 0$, then $|\xi_x| > 0$ which would give a non-trapped geodesic. Then, the Hamiltonian becomes $H(x, \theta_1, \xi_x, \xi_{\theta_1}) = \xi_x^2 + A_1^{-2}$. Notice that the value of $\theta_1$ does not change the Hamiltonian, so we can plot the Hamiltonian as level sets in the variables $x$ and $\xi_x$.

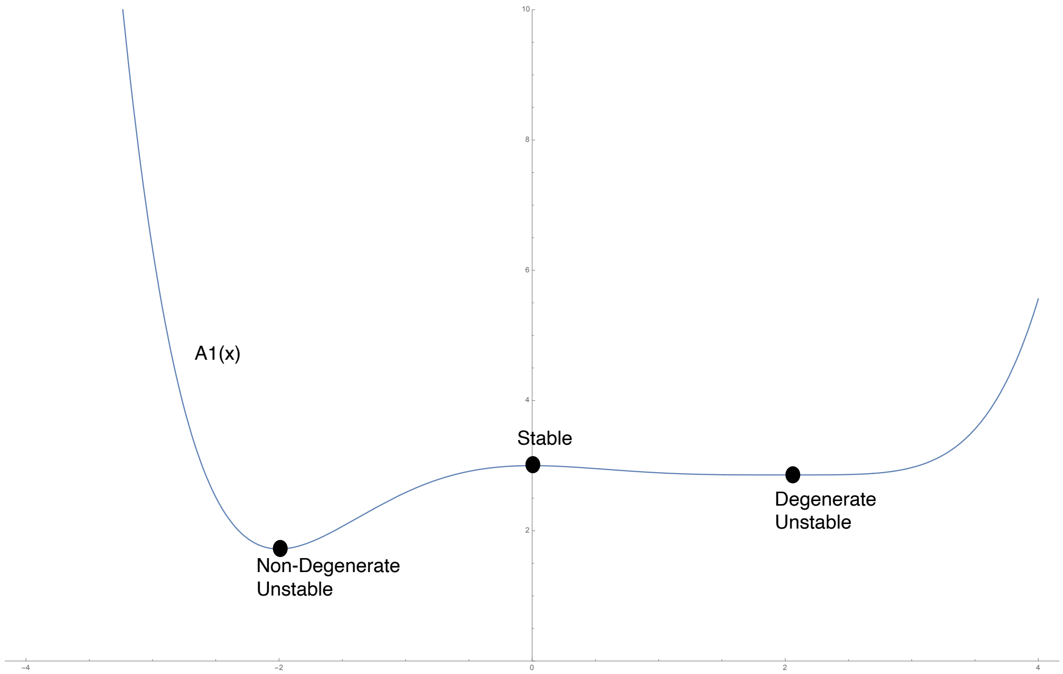


Figure 2.2: Graph of $A_1(x)$

Consider $A_1$ given in Figure 2.2, which has a local minimum at $x = -2$, a local maximum at $x = 0$ and a local minimum at $x = 2$ such that $A_1''(2) = A_1'''(2) = 0$. Note that the Hamiltonian is determined by $A_1^{-2}$. The point $x = -2$ will have non-degenerate unstable trapping, the point $x = 0$ has stable trapping and the point $x = -2$ will have degenerate unstable trapping. We can observe this behavior in the plot of the level sets of the Hamiltonian given in Figure 2.3.

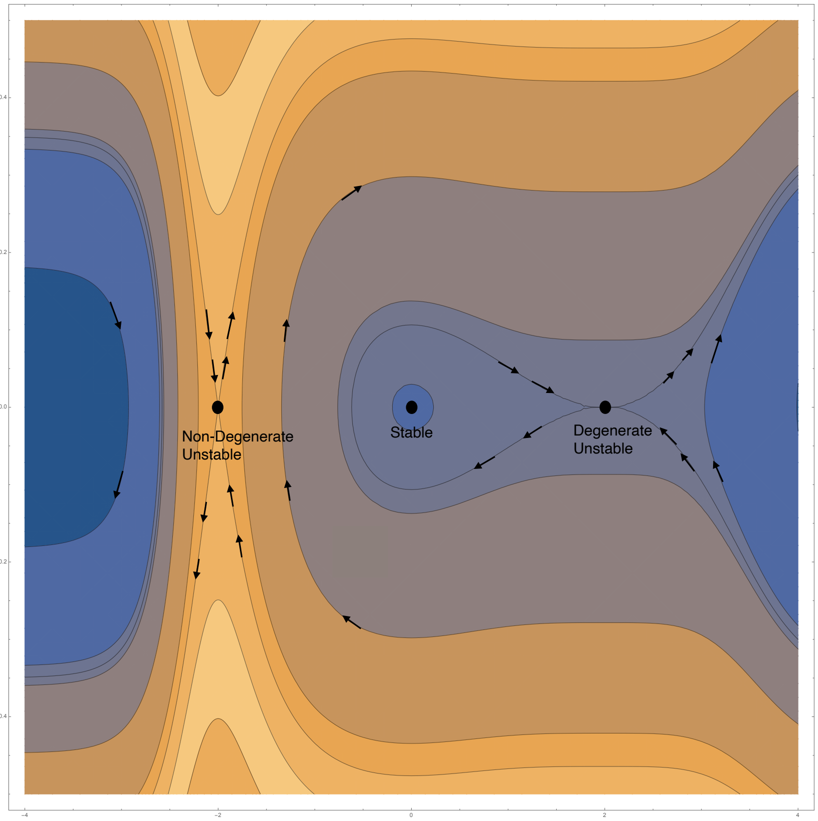


Figure 2.3: Level Sets of the Hamiltonian

Notice that at $x = -2$ if $\xi_x \neq 0$, then the trajectory will go to $\pm\infty$. Similarly, if we start at a point near $x = -2$, but not equal to $x = -2$ the trajectory will go to $\pm\infty$ even if $\xi_x(0) = 0$. This is an example of unstable trapping at $x = -2$. Notice that near $x = 0$ level sets are closed curves. This means near $x = 0$, every ray will be trapped. If $(x, \xi_x)$ is within a small neighborhood of $(0, 0)$, then the trajectory with those initial conditions is still trapped. This is an example of stable trapping at $x = 0$. Note that near $x = 2$ the level sets are similar to those at $x = -2$. We have unstable behavior. However, the stable and unstable manifolds approach the point $x = 2$, $\xi_x = 0$ tangential to each other. This is a way to see degenerate trapping. The issue with this is that it precludes any sort of normal form and that the trajectories near the degenerate point will move towards infinity at a slower rate.

**Remark 2.2.3.** *We discussed the trapping in terms of $x$, since the value of $\theta_1$ does not* *change the Hamiltonian. However, if a geodesics $\gamma(t) = (x(t), \theta_1(t))$ is trapped with initial condition $x(0) = x_0, \theta_1(0) = \theta_0, \dot{x}(0) = \xi_x, \dot{\theta}_1(0) = \xi_{\theta_1}$, then the geodesic $\tilde{\gamma}(t) = (\tilde{x}(t), \tilde{\theta}_1(t))$ with initial condition $x(0) = x_0, \theta_1(0) = \tilde{\theta}_0, \dot{x}(0) = \xi_x, \dot{\theta}_1(0) = \xi_{\theta_1}$ is trapped for any $\tilde{\theta}_0 \in \mathbb{S}^1$.*

Now we will consider the multi-warped product case. Let $M = \mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{S}^1$ with a metric

$$g = dx^2 + A_1^2(x)d\theta_1^2 + A_2^2(x)d\theta_2^2$$

where $A_1(x), A_2(x) > 0$, $A_1(x)$ has a critical point at $x = 1$, $A_2(x)$ has a critical point at $x = 2$ and $\partial_x(A_1^{-2}), \partial_x(A_2^{-2}) \le 0$. In this situation

$$\Gamma^x_{\theta_1\theta_1} = -A_1A_1',\ \Gamma^x_{\theta_2\theta_2} = -A_2A_2',\ \Gamma^x_{ij} = 0 \text{ for all other } i, j.$$

Let $\gamma(t) = (x(t), \theta_1(t), \theta_2(t))$ be a geodesic on $M$. Then,

$$\frac{d^2x(t)}{dt^2} - A_1A_1'\left(\frac{d\theta_1}{dt}\right)^2 - A_2A_2'\left(\frac{d\theta_2}{dt}\right)^2 = 0.$$

In terms of the Hamiltonian flow

$$\begin{aligned}
\dot{x} &= \xi_x \\
\dot{\theta}_1 &= A_1^{-2}\xi_{\theta_1} \\
\dot{\theta}_2 &= A_2^{-2}\xi_{\theta_2} \\
\dot{\xi}_x &= -\frac{1}{2}(\partial_x(A_1^{-2})\xi_{\theta_1}^2 + \partial_x(A_2^{-2})\xi_{\theta_2}^2) \\
\dot{\xi}_{\theta_1} &= 0 \\
\dot{\xi}_{\theta_2} &= 0.
\end{aligned}$$

Notice that $\dot{\xi}_x \ge 0$. If the Hamiltonian flow has initial condition $\xi_x(0) > 0$, then the trajectory is not trapped as $t \to \infty$. Furthermore, if $\xi_x(0) < 0$, then the trajectory is not trapped as $t \to -\infty$. This implies the only possible trapped trajectories must have initial condition $\xi_x(0) = 0$. If both $\xi_{\theta_1}, \xi_{\theta_2} > 0$, we know $\dot{\xi}_x(0) > 0$, since the critical points for $A_1^{-2}$ and $A_2^{-2}$ are at different $x$-values. Hence, any trajectory with initial condition $\xi_{\theta_1}(0), \xi_{\theta_2}(0) > 0$ will not be trapped. Assume that $\xi_{\theta_2}(0) = 0$. Then, $\dot{\xi}_x(0) > 0$ as long as $x \neq 1$. This implies that the only trapped trajectories with initial conditions $\xi_{\theta_1}(0) \neq 0, \xi_{\theta_2}(0) = 0, \xi_x(0) = 0$ have the initial condition $x(0) = 1$. Similarly, the only trapped trajectories with initial conditions $\xi_{\theta_1}(0) = 0, \xi_{\theta_2}(0) \neq 0, \xi_x(0) = 0$ will have the initial condition $x(0) = 2$. This analysis shows that the trapping only occurs at $x = 1$ and $x = 2$ and it is isolated to the $\theta_1$-direction at $x = 1$ and the $\theta_2$-direction at $x = 2$. This is the trapping in [CN22].

Now let $M = \mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$ with a metric

$$g = dx^2 + A_1^2(x)d\theta_1^2 + A_2^2(x)d\theta_2^2$$

where $A_1(x), A_2(x) > 0$, $A_1(x)$ has a critical point at $x = -1$, and $A_2(x)$ has a critical point at $x = 1$. Assume $\partial_x(A_1^{-2}) > 0$ for $x < -1$ and $\partial_x(A_1^{-2}) < 0$ for $x > -1$. Assume $\partial_x(A_1^{-2}) > 0$ for $x < 1$ and $\partial_x(A_1^{-2}) < 0$ for $x > 1$. Notice that for $x \in (-1, 1)$ we have that $\text{sign}(A_1'(x)) \neq \text{sign}(A_2'(x))$ and that for $x \notin [-1, 1]$ we have $\text{sign}(A_1'(x)) = \text{sign}(A_2'(x))$. Suppose $\xi_x(0) = 0$ and $x(0) \in (-1, 1)$ for initial conditions of for the Hamiltonian flow. Since $\text{sign}(A_1'(x)) \neq \text{sign}(A_2'(x))$, we have that there exists initial conditions for $\xi_{\theta_1}$ and $\xi_{\theta_2}$ such that $\dot{\xi}_x(0) = 0$. Hence, there will be trapped trajectories for the initial conditions $\xi_x(0) = 0$ and $x(0) \in (-1, 1)$. This situation is what makes the trapping in this thesis different than the case in [CN22]. Instead of being isolated to two points in $x$, the trapping is on a range of $x$ values. Additionally, a trapped trajectory can have non-zero velocities in both the $\theta_1$ and $\theta_2$ directions, unlike the manifold in [CN22].

## 2.3 Pseudo-differential Operators

### 2.3.1 Classical Pseudo-differential Operators

We will be interested in pseudo-differential operators which are a class of integral operators. The main class of functions we will apply pseudo-differential operators to are Schwartz functions.

**Definition 2.3.1.** *Let*

$$\mathscr{S}(\mathbb{R}^n) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^n) \middle| \forall \alpha, \beta \in \mathbb{N}^n, \sup_{x \in \mathbb{R}^n} |x^\alpha D_x^\beta f(x)| < \infty \right\}$$

*denote the space of Schwartz functions on* $\mathbb{R}^n$.

This class of functions is nice to work with because the integral operators we are interested in are defined from $\mathscr{S}$ to $\mathscr{S}$ and $\mathscr{S}$ functions are dense in $L^2$ functions.

For a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$

$$D_x^\alpha u(x) = \frac{1}{i^{|\alpha|}} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \cdots D_{x_n}^{\alpha_n} u(x_1, x_2, \cdots, x_n)$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Note that

$$D_x^\alpha u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \xi^\alpha \hat{u}(\xi) d\xi$$

for any multi-index $\alpha$. In general, given a differential operator $L = \sum_{|\alpha| \le k} l_\alpha(x) D_x^\alpha$, define $a(x, \xi) = \sum_{|\alpha| \le k} l_\alpha(x) \xi^\alpha$. Then,

$$Lu(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} a(x, \xi) \hat{u}(\xi) d\xi.$$

We call $a(x, \xi)$ the symbol of the operator $L$. We want to allow a wider class of functions for $a(x, \xi)$ than just polynomials in $\xi$; however, we would like the functions to behave similarly to polynomials as $|\xi| \to \infty$. This is the idea behind the definition of symbol classes.

**Definition 2.3.2.** *Let $m$ be a real number. The symbol class of order $m$ on $\mathbb{R}^n$ is the set*

$$S^m(\mathbb{R}^n) = \{a(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n) | \, |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|} \text{ for all multindices } \alpha, \beta\}$$

*where $\langle \xi \rangle^s = (1+\xi^2)^{s/2}$.*

**Definition 2.3.3.** *For $a \in \mathcal{C}^\infty(\mathbb{R}^{2n})$, $a = \mathcal{O}_{S^m}(1)$ if*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|} \text{ for all multindices } \alpha, \beta.$$

**Theorem 2.3.4** (Theorem 18.1.6 in [Hör07])**.** *If $a \in S^m$ and $u \in \mathscr{S}$, then*

$$a(x,D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi \rangle} a(x,\xi) \hat{u}(\xi) d\xi$$

*defines a function $a(x,D)u \in \mathscr{S}$, and the bilinear map $(a,u) \mapsto a(x,D)u$ is continuous. One calls $a(x,D)$ or $\operatorname{Op}^1(a)$ a pseudo-differential operator of order $m$. We will call $a(x,D)$ the standard quantization of $a(x,\xi)$.*

**Theorem 2.3.5** (Theorem 18.1.8 in [Hör07])**.** *If $a_j \in S^{m_j}$, $j = 1,2$, then as operators on $\mathscr{S}$*

$$a_1(x,D)a_2(x,D) = b(x,D)$$

*where $b \in S^{m_1+m_2}$ is given by*

$$b(x,\xi) = e^{i\langle D_y, D_\eta \rangle} a_1(x,\eta) a_2(y,\xi)|_{\eta=\xi, y=x}. \tag{2.3.6}$$

If we calculate out terms in (2.3.6) we get

$$\begin{aligned} b(x,\xi) &= e^{i\langle D_y, D_\eta\rangle} a_1(x,\eta) a_2(y,\xi)|_{\eta=\xi, y=x} \\ &= a_1(x,\xi)a_2(x,\xi) + \frac{1}{2} D_x a_1(x,\xi) D_\xi a_2(x,\xi) + \mathcal{O}_{S^{m_1+m_2-2}}(1). \end{aligned}$$

Since $a_1(x,\xi)a_2(x,\xi) = a_2(x,\xi)a_1(x,\xi)$, we get the following corollary,

**Corollary 2.3.7.** *If $a_j \in S^{m_j}$, $j = 1, 2$, then as operators on $\mathscr{S}$ we have $[a_1(x,D), a_2(x,D)] = b(x,D)$ where $b \in S^{m_1+m_2-1}$.*

If $a_1$ or $a_2$ is a polynomial in $x$ and $\xi$, there are only a finite number of terms in the expansion for $b(x,\xi)$.

**Theorem 2.3.8** (Theorem 18.1.9 in [Hör07])**.** *If $a \in S^0$, then $a(x,D)$ is bounded in $L^2(\mathbb{R}^n)$.*

**Corollary 2.3.9.** *If $a \in S^m$, then $a(x,D)$ is a continuous operator from $H^s$ to $H^{s-m}$.*

*Proof.* Suppose $u \in H^s(\mathbb{R}^n)$. $||u||_{H^s} = ||\Lambda^s u||_{L^2}$ where $\Lambda^s u(x) = \langle D_x\rangle^s u(x)$. Then,

$$||a(x,D)u||_{H^{s-m}} = ||\Lambda^{s-m} a(x,D)u||_{L^2} = ||\Lambda^{s-m}\Lambda^m\Lambda^{-m} a(x,D)u||_{L^2} = ||\Lambda^s\Lambda^{-m} a(x,D)u||_{L^2}.$$

From Theorem 2.3.5, $\Lambda^{-m}a(x,D) = a_1(x,D)$ for a symbol $a_1 \in S^0$. Using Theorem 2.3.8 gives

$$||\Lambda^s(a_1(x,D)u)||_{L^2} \leq C||\Lambda^s u||_{L^2} = C||u||_{H^s}$$

for a constant $C > 0$. □

**Lemma 2.3.10.** *Let $a(x,\xi) \in S^s(\mathbb{R}^n)$ be a symbol only dependent on $\xi$. Let $f, g \in H^s$. Then,*

$$\langle a(x,D)f, g\rangle = \langle f, \bar{a}(x,D)g\rangle.$$

This is a corollary of Proposition 2.3.20

### 2.3.2 $h$-calculus

We will begin by developing the classical $h$-calculus and Weyl quantization as done in [Zwo12] and then explain why there is a need for the two-parameter calculus. Note that we will require $0 < h \leq 1$. We will use the following symbol class for the $h$-calculus,

$$\mathcal{S}_\delta(\mathbb{R}^n) := \big\{a \in \mathcal{C}^\infty(\mathbb{R}^{2n}) \big| \, |\partial^\alpha a| \leq C_\alpha h^{-\delta|\alpha|} \text{ for all multi-indices } \alpha\big\}. \tag{2.3.11}$$

**Definition 2.3.12.** *For $a \in \mathcal{C}^\infty(\mathbb{R}^{2n})$, $a = O_{\mathcal{S}_\delta}(h^N)$ if $|\partial^\alpha a| \leq C_{\alpha,N} h^{N-\delta|\alpha|}$ for all multi-indices $\alpha$.*

Let $\tilde{x} := h^{-1/2}x, \tilde{\xi} := h^{-1/2}\xi$ and $a_h(\tilde{x}, \tilde{\xi}) := a(x, \xi)$. If $a \in \mathcal{S}_\delta$, then

$$|\partial^\alpha a_h| = h^{|\alpha|/2} |\partial^\alpha a| \leq C_\alpha h^{|\alpha|(\frac{1}{2}-\delta)}.$$

Under this scaling we see that if $\delta = \frac{1}{2}$, there is no decay as $h \to 0$. There are two issues we want to work around here. We will want to look at the critical case to minimize gain in orders of $h$ as we take derivatives. Additionally, we will want to make this more general, so that there is different decay depending on derivatives in $x$ or $\xi$. To handle the difference in decay depending on derivatives in $x$ or $\xi$ we will use the scaling $\tilde{x} := h^{-\beta}x, \tilde{\xi} := h^{-\gamma}\xi$, where $\beta + \gamma = 1$. This will give a similar, but more general critical case where there is no decay as $h \to 0$. We will then introduce a second parameter, so that we can handle these critical cases.

Now that we defined symbol classes, we can introduce the Weyl quantization and semi-classical standard quantization of a symbol.

**Definition 2.3.13** (Quantization). *The Weyl quantization of the symbol $a \in \mathcal{S}_\delta$ is the operator denoted $a^w(x, hD)$ or $\mathrm{Op}_h^w(a)$ acting on a function $u \in \mathscr{S}(\mathbb{R}^n)$ by the formula*

$$a^w(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\Big(\frac{x+y}{2}, \xi\Big) u(y) dy d\xi. \tag{2.3.14}$$

*The semi-classical standard quantization is the operator denoted by* $a(x, hD)$ *or* $\mathrm{Op}\,_h^1(a)$ *by the formula*

$$a^w(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\big(x, \xi\big) u(y) dy d\xi. \tag{2.3.15}$$

*More generally for* $0 \leq t \leq 1$ *we define the operator* $\mathrm{Op}\,_h^t(a)$ *by*

$$a^w(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\big(tx + (1-t)y, \xi\big) u(y) dy d\xi. \tag{2.3.16}$$

Note that $\mathrm{Op}\,_h^1(a) = \mathrm{Op}\,^1(a)$ for $h = 1$.

**Theorem 2.3.17** (Theorem 4.13 from [Zwo12])**.** *If* $A = Op_h^t(a_t)$ *for* $0 \leq t \leq 1$*, then*

$$a_t(x, \xi) = e^{i(t-s)h\langle D_x, D_\xi\rangle} a_s(x, \xi).$$

This gives us a formula to change quantizations. From the change of quantization formula, we should expect that the Weyl quantization has many of the same properties as the standard quantization.

**Theorem 2.3.18** (Theorem 4.16 from [Zwo12])**.** *If* $a \in \mathcal{S}_\delta$*, then*

$$a^w(x, hD) : \mathscr{S} \to \mathscr{S}$$

*and*

$$a^w(x, hD) : \mathscr{S}' \to \mathscr{S}'$$

*are continuous linear transformations.*

**Theorem 2.3.19** (Theorem 4.23 [Zwo12])**.** *Let* $a \in \mathcal{S}$*, then*

$$a^w : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

*is bounded.*

We have the following generalization of Lemma 2.3.10.

**Proposition 2.3.20.** *Let $a \in \mathcal{S}$. Then,*

$$\operatorname{Op}_t^*(a) = \operatorname{Op}_{1-t}(\bar{a})$$

*for $(0 \leq t \leq 1)$ and in particular if $a$ is real then*

$$a^w(x, hD)^* = a^w(x, hD).$$

*Proof.* Let $f, g \in L^2$. Then,

$$\begin{aligned}
\langle \operatorname{Op}_t(a) f, g\rangle &= \int_{\mathbb{R}^n} \operatorname{Op}_t(a) f(x) \bar{g}(x) dx \\
&= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} a(tx + (1-t)z, \xi) e^{-i\langle z, \xi\rangle} f(z) dz d\xi \bar{g}(x) dx \\
&= \int_{\mathbb{R}^n} f(z) \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} a(tx + (1-t)z, \xi) e^{-i\langle z, \xi\rangle} \bar{g}(x) dx d\xi dz \\
&= \int_{\mathbb{R}^n} f(z) \frac{1}{(2\pi)^n} \overline{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle z, \xi\rangle} \bar{a}(tx + (1-t)z, \xi) e^{-i\langle x, \xi\rangle} g(x) dx d\xi} dz \\
&= \int_{\mathbb{R}^n} f(z) \overline{\operatorname{Op}_{1-t}(\bar{a}) g(z)} dz \\
&= \langle f, \operatorname{Op}_{1-t}(\bar{a})(x, D) g\rangle.
\end{aligned}$$

□

Now, that we know $a^w(x, hD)$ is a well-defined operator, we would like to know what happens when we take the composition of two quantizations.

**Theorem 2.3.21** (Theorem 4.18 from [Zwo12])**.** *Suppose $a \in \mathcal{S}_\delta$ and $b \in \mathcal{S}_\delta$. Let*

$$a \# b(x, \xi) := e^{ihA(D)}(a(x, \xi) b(x, \xi))|_{y=x, \eta=\xi}.$$

*$A(D) := \frac{1}{2}\sigma(D_x, D_\xi, D_y, D_\eta)$. Then, $a\#b \in \mathcal{S}_\delta$ and*

$$a^w(x, hD)b^w(x, hD) = (a\#b)^w(x, hD)$$

*as operators mapping $\mathscr{S}$ to $\mathscr{S}$. Furthermore,*

$$a\#b = ab + \frac{h}{2i}\{a, b\}(x, \xi) + \mathcal{O}_{\mathcal{S}_\delta}(h^{2-2\delta})$$

*and*

$$[a^w(x, hD), b^w(x, hD)] = \frac{h}{i}\{a, b\}^w(x, hD) + \mathcal{O}_{\mathcal{S}_\delta}(h^{3(1-2\delta)}). \tag{2.3.22}$$

**Remark 2.3.23.** *There are a few reasons why the Weyl quantization is useful. One reason is that the Weyl quantization is essentially self-adjoint with respect to the $L^2$ inner product if the symbol is real. This is not the case for standard quantization. Additionally, in the commutator* (2.3.22) *the terms with even order of derivatives cancel out leaving only the odd terms. This is why we get $\mathcal{O}_{\mathcal{S}_\delta}(h^{3(1-2\delta)})$ in* (2.3.22) *instead of $\mathcal{O}_{\mathcal{S}_\delta}(h^{2(1-2\delta)})$ for the semi-classical standard quantization.*

We will need a similar result to this to prove the main results of this thesis. The issue is that we are dealing with a critical case, which means that the $\mathcal{O}_{\mathcal{S}_\delta}(h^{3(1-\delta)})$ term turns out to be $\mathcal{O}_{\mathcal{S}_\delta}(h^0)$. Since, there is no gain in $h$ we cannot easily absorb the terms in the expansion into the Poisson bracket term. This is why we introduce a second small parameter that is gained, so that we can absorb the additional terms in the critical case.

### 2.3.3 Two parameter calculus

In this section we will introduce the two parameter calculus used to handle the marginal $h$-calculus situations. This two parameter calculus was first introduced by Sjöstrand and Zworski in [SZ07] for the case of $\alpha = \beta = 1/2$. Then, the two parameter calculus was expanded to allow for $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ in [CW13].

To start we need to define a new symbol class. For $\alpha \in [0,1]$ and $\beta \leq 1-\alpha$, let

$$\mathcal{S}^{k,m,\tilde{m}}_{\alpha,\beta}(\mathbb{R}^n) := \left\{ a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times (0,1]^2) \,\Big|\, |\partial_x^\rho \partial_\xi^\gamma a(x,\xi;h,\tilde{h})| \leq C_{\rho\gamma} h^{-m} \tilde{h}^{-\tilde{m}} \Big(\frac{\tilde{h}}{h}\Big)^{\alpha|\rho|+\beta|\gamma|} \langle \xi \rangle^{k-|\gamma|} \right\}.$$

Throughout this paper we will assume that $\tilde{h} \geq h$. Notice that if $\alpha, \beta = 1/2$, then we are in the critical case, where there is no gain in $h$ when computing the terms of the composition of two quantizations. However, we see that there are gains in $\tilde{h}$, which will allow us to handle the additional terms beyond the Poisson bracket teerm in the expansion for the commutator of two pseudo-differential operators. In this thesis we will specifically consider the case where $\beta = \frac{2}{3}$ and $\alpha = \frac{1}{3}$.

If $a \in \mathcal{S}^{k,m,\tilde{m}}_{\alpha,\beta}$ and $b \in \mathcal{S}^{k,m',\tilde{m}'}_{\alpha,\beta}$, then

$$\mathrm{Op}_h^w(a) \circ \mathrm{Op}_h^w(b) = \mathrm{Op}_h^w(c) \text{ with } c \in \mathcal{S}^{k+k',m+m',\tilde{m}+\tilde{m}'}_{\alpha,\beta}.$$

Additionally, we have the following lemma from [CW13],

**Lemma 2.3.24** (Lemma 2.4 from [CW13])**.** *Suppose that $a, b \in \mathcal{S}^{0,0,0}_{\alpha,\beta}$ and that $c^w = a^w \circ b^w$. Then*

$$c(x,\xi) = \sum_{k=0}^{N} \frac{1}{k!} \Big(\frac{ih}{2} A(D)\Big)^k a(x,\xi) b(y,\eta)|_{x=y,\xi=\eta} + e_N(x,\xi)$$

*where for some $M$*

$$|\partial^\gamma e_N| \leq C_N h^{N+1} \sum_{\gamma_1+\gamma_2=\gamma} \sup_{(x,\xi)\in T^*\mathbb{R}^n,(y,\eta)\in T^*\mathbb{R}^n} \sup_{|\rho|\leq M, \rho\in\mathbb{N}^{4n}} \left| \Gamma_{\alpha,\beta,\rho,\gamma}(D)(A(D))^{N+1} a(x,\xi) b(y,\eta) \right|$$

*where*

$$\Gamma_{\alpha,\beta,\rho,\gamma}(D) = (h^\alpha \partial_{(x,y)}, h^\beta \partial_{(\xi,\eta)})^\rho \partial^{\gamma_1}_{(x,\xi)} \partial^{\gamma_2}_{(y,\eta)}.$$

Notice that if $a \in \mathcal{S}^{0,0,0}_{\alpha,\beta}(\mathbb{R}^n)$ and $b \in \mathcal{S}(\mathbb{R}^n)$ then,

$$c(x,\xi) = \sum_{k=0}^{N} \frac{1}{k!}\Big(\frac{ih}{2}A(D)\Big)^k a(x,\xi)b(y,\eta)|_{x=y,\xi=\eta}$$
$$+ \mathcal{O}_{\mathcal{S}^{0,0,0}_{\alpha,\beta}}(h^{N+1}\max\{(\tilde{h}/h)^{(N+1)\alpha}, (\tilde{h}/h)^{(N+1)\beta}\}). \tag{2.3.25}$$

This gain in $\tilde{h}$ terms allows us to absorb the high order terms that we could not absorb in the past due to the lack of gain of powers of $h$.

## 2.4 Functional Calculus

We will review functional calculus, so that we can define the fractional Laplacian, the Schrödinger propagator and Sobolev spaces on a manifold $H^s(M)$ for non-integer $s$. We will follow the results presented in [RS81] and [LPG+19].

**Theorem 2.4.1** (Theorem VIII.5 in [RS81])**.** *Let $A$ be a self-adjoint operator on a Hilbert space $H$. Then, there is a unique map $\hat{\varphi}$ from bounded borel functions on $\mathbb{R}$ to linear operators on $H$, $\mathcal{L}(H)$ so that*

*a) $\hat{\varphi}$ is an algebraic $*$-homomorphism*

*b) $\hat{\varphi}$ is norm continuous, that is $||\hat{\varphi}(h)||_{\mathcal{L}(H)} \leq ||h||_\infty$*

*c) If $A\psi = \lambda\psi$, then $\hat{\varphi}(f)\psi = f(\lambda)\psi$.*

This theorem allows use to define $e^{it\Delta}$ used in 4.4.2 since $f(x) = e^{itx}$ is a bounded Borel function on $\mathbb{R}$. The functional calculus form is useful, however $f(x) = x^s$ for $0 < s < 1$ is not a bounded function. Theorem 2.4.1 does not allow us to construct the fractional Laplacian in this form. We will have to use another form of the spectral theorem.

To start we will define projection-valued measures.

**Definition 2.4.2.** *Let $P_\Omega$ be the operator $\chi_\Omega(A)$ where $\chi_\Omega$ is the characteristic function on the measurable set $\Omega \subset \mathbb{R}$. Suppose the family of operators $\{P_\Omega\}$ has the following properties:*

*a) Each $P_\Omega$ is an orthogonal projection*

*b) $P_\emptyset = 0, P_{(-\infty,\infty)} = I$*

*c) If $\Omega = \cup_{n=1}^{N} \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ if $n \neq m$, then $P_\Omega = s - \lim_{N\to\infty} \sum_{n=1}^{N} P_{\omega_n}$*

*d) $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$*

*Then, the family is called a projection-valued measure.*

Fix $A$, an unbounded self-adjoint operator on a Hilbert space $H$. Let $P_\Omega$ be the operator $\chi_\Omega(A)$. From Theorem 2.4.1 we know this is a well-defined operator. Let $\psi \in H$, then $\langle \psi, P_\Omega \psi \rangle$ is a well-defined Borel measure on $\mathbb{R}$ which we denote by $d(\psi, P_\lambda \psi)$. In particular $\{P_\Omega\}$ is a projection-valued measure.

Given a bounded Borel function $g$ we can define $g(A)$ by

$$\langle \psi, g(A)\psi \rangle = \int_{-\infty}^{\infty} g(\lambda) d(\psi, P_\lambda \psi).$$

**Theorem 2.4.3** (Theorem VIII.6 in [RS81])**.** *There is a one-to-one correspondence between self-adjoint operators $A$ and projection-valued measures $\{P_\Omega\}$ on $H$, the correspondence being given by*

$$A = \int_{-\infty}^{\infty} \lambda d(\psi, P_\lambda \psi)$$

This construction is more general than the functional calculus form. It allows us to define $A^\alpha$ for $0 < \alpha < 1$ for a positive operator $A$.

**Definition 2.4.4.**

$$A^\alpha = \int_{-\infty}^{\infty} \lambda^\alpha d(\psi, P_\lambda \psi)$$

Taking $A = -\Delta_g$ we can define non-integer powers of the Laplacian operator on a manifold. Specifically,

**Definition 2.4.5.**

$$(-\Delta)^\alpha = \int_{-\infty}^{\infty} \lambda^\alpha d(\psi, P_\lambda \psi)$$

We take $-\Delta_g$ instead of $\Delta_g$, because $-\Delta_g$ is a positive operator and the spectrum of $-\Delta_g$ is contained in the positive real line. Thus,

$$(-\Delta)^\alpha = \int_0^\infty \lambda^\alpha d(\psi, P_\lambda \psi)$$

As an example we will consider the case of $A = -\Delta$ on $\mathbb{R}^n$. Note that the spectrum of $-\Delta$, denoted by $\sigma(-\Delta)$, consists of $|\xi|^2$, where $\xi \in \mathbb{R}$ with corresponding eigenfuctions $e^{-i\xi\cdot x}$. Recall we use $-\Delta$, so that the spectrum is positive. Thus the projection valued measure is given by

$$d(\psi, P_\xi \psi) = \frac{1}{(2\pi)^d}\langle \psi, e^{i\xi, x}\rangle e^{i\xi\cdot x} d\xi.$$

So,

$$(-\Delta)^{\alpha/2}u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^d} |\xi|^\alpha \langle u, e^{i\xi, x}\rangle e^{i\xi\cdot x} d\xi = F^{-1}(|\xi|^\alpha F(u))(x).$$

This shows how in $\mathbb{R}^n$, $(-\Delta)^{s/2}u = \mathrm{Op}^1(|\xi|^s)u$. In general, if we have positive self-adjoint operators $A$ and $B$ such that $A^s = B^s$, then we know that $A = B$.

Now to define the norm for $H^s(M)$ we will take

$$||u||_{H^s(M)} = ||(C-\Delta)^{s/2}u||_{L^2(M)}$$

for a constant $C > 0$ sufficiently large. Note that

$$||(C-\Delta)^{s/2}u||_{L^2(M)} \geq C^{s/2}||u||_{L^2} + ||(-\Delta)^{s/2}u||_{L^2(M)}$$

so that this provides an equilvalent metric to the $H^s(M)$ norm defined through pseudo-differential calculus. In fact, on $\mathbb{R}^n$

$$(1-\Delta)^{s/2}u = \langle D_x\rangle^s u.$$

## 2.5 Schrödinger Equation

Let $M$ be a Riemannian manifold without boundary with metric $g$ and Laplace-Beltrami operator $\Delta_g$. The Schrödinger equation on $M$ is

$$\begin{cases} (D_t - \Delta_g)u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

where $D_t = \frac{1}{i}\partial_t$.

The Schrödinger equation has the property that the $H^s(\mathbb{R}^n)$ norm is preserved. This implies that if the initial condition $u_0 \in H^s(\mathbb{R}^n)$ for some $s > 0$, but $u_0 \notin H^{s+\delta}(\mathbb{R}^n)$ for some small $\delta > 0$, then $u(t,\cdot) \in H^s(\mathbb{R}^n)$ and $u(t,\cdot) \notin H^{s+\delta}(\mathbb{R}^n)$ for fixed $t$. This implies $u$ does not become more well behaved globally in $x$ over time. Additionally, it is possible for $u$ to concentrate locally in $x$ at a single point in time. For example, an estimate of the form $||e^{it\Delta}u_0||^2_{H^{1/2}_{loc}} \leq C||u_0||^2_{L^2}$ is not possible for all $t > 0$ if $u_0 \notin H^{1/2}$. Both of these issues are why the best we can hope for is local smoothing averaged over time. We will end this section by showing conservation of the $H^s$ norm for Schrödinger equation before we move onto local smoothing results in the later sections.

**Lemma 2.5.1.** *Let $u_0 \in H^s(\mathbb{R}^n)$. Suppose $u$ is the solution to*

$$\begin{cases} (D_t - \Delta)u = 0 \\ u|_{t=0} = u_0. \end{cases}$$

*Then*

$$||u(t,-)||_{H^s(\mathbb{R}^n)} = ||u_0||_{H^s(\mathbb{R}^n)}$$

*for all $t \geq 0$.*

*Proof.* We will prove this for $u_0 \in \mathscr{S}$. Since Schwartz functions are dense in $H^s$, we get the results for all functions $u_0 \in H^s$. Additionally, if $u_0 \in \mathscr{S}$, then $\hat{u_0} \in \mathscr{S}$.

Suppose $u_0 \in \mathscr{S}$. If we take the Fourier transform in $x$ of the Schrödinger Equation we get

$$\begin{cases} (D_t + |\xi|^2)\hat{u} = 0 \\ \hat{u}|_{t=0} = \hat{u}_0. \end{cases}$$

This has the solution $\hat{u} = e^{-i|\xi|^2 t}\hat{u}_0$. Now, taking the inverse Fourier transform gives that

$$u(t,x) = e^{it\Delta}u_0(x) = \mathscr{F}^{-1}(e^{-i|\xi|^2 t}\hat{u}_0)$$

is the solution to the Schrödinger equation. The operator $e^{it\Delta}$ is called the Schrödinger propagator.

Now recall that

$$||u||_{H^s} = ||\Lambda^s u||_{L^2}$$

where

$$\Lambda^s u = \mathscr{F}^{-1}(\langle\xi\rangle^s \hat{u}).$$

Using that $u(t,x) = e^{it\Delta}u_0(x)$ and that $|e^{-it|\xi|^2}| = 1$ gives

$$\begin{aligned} ||u||_{H^s} &= ||\Lambda^s e^{it\Delta}u_0||_{L^2} \\ &= \frac{1}{(2\pi)^n}||\langle\xi\rangle^s e^{it|\xi|^2}\hat{u}_0||_{L^2} \\ &= \frac{1}{(2\pi)^n}||\langle\xi\rangle^s \hat{u}_0||_{L^2} \\ &= ||\Lambda^s u_0||_{L^2} \\ &= ||u_0||_{H^s}. \end{aligned}$$

□

We will now consider the Schrödinger equation with a potential term $V(x)$ where $V \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $V$ and all of its derivatives are bounded. The addition of the $V(x)$ term here does not allow us to use the same proof as in the case of $\mathbb{R}^n$ without a potential term. Let $u$ be a solution to

$$\begin{cases} (D_t - \Delta + V)u = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{2.5.2}$$

and $u_0 \in \mathscr{S}(\mathbb{R}^n)$. Let $E(t) = ||u(t,\cdot)||^2_{L^2(\mathbb{R}^n)}$. Then,

$$\begin{aligned} E'(t) &= 2\operatorname{Re} i \int_{\mathbb{R}^n} (D_t u)\bar{u}dx \\ &= 2\operatorname{Re} i \int_{\mathbb{R}^n} (\Delta - V)u\bar{u}dx \\ &= 2\operatorname{Re} i \int_{\mathbb{R}^n} (\Delta u)\bar{u}dx - 2\operatorname{Re} i \int_{\mathbb{R}^n} (-Vu)\bar{u}dx \\ &= 2\operatorname{Re} i \int_{\mathbb{R}^n} (\Delta u)\bar{u}dx, \text{ since } -Vu\bar{u} \text{ is real} \\ &= -2\operatorname{Re} i \int_{\mathbb{R}^n} |\nabla u|^2 dx \\ &= 0, \text{ since } |\nabla u|^2 \text{ is real.} \end{aligned}$$

Hence, $E(t) = E(0)$ for all $t$.

Let $E_1(t) = ||\nabla u(t,\cdot)||^2_{L^2(\mathbb{R}^n)}$. Then,

$$\begin{aligned}
E_1'(t) &= 2\operatorname{Re} i\int_{\mathbb{R}^n}(D_t\nabla u)\cdot\nabla\bar{u}dx \\
&= 2\operatorname{Re} i\int_{\mathbb{R}^n}\nabla((\Delta - V)u)\cdot\nabla\bar{u}dx \\
&= 2\operatorname{Re} i\int_{\mathbb{R}^n}\nabla(\Delta u)\cdot\nabla\bar{u}dx - 2\operatorname{Re} i\int_{\mathbb{R}^n}\nabla(-Vu)\cdot\nabla\bar{u}dx \\
&= 2\operatorname{Re} i\int_{\mathbb{R}^n}-|\Delta u|^2dx - 2\operatorname{Re} i\int_{\mathbb{R}^n}\nabla(-Vu)\cdot\nabla\bar{u}dx \\
&= 2\operatorname{Re} i\int_{\mathbb{R}^n}u\nabla(-V)\cdot\nabla\bar{u}dx - 2\operatorname{Re} i\int_{\mathbb{R}^n}-V\nabla u\cdot\nabla\bar{u}dx \\
&= 2\operatorname{Re} i\int_{\mathbb{R}^n}u\nabla(-V)\cdot\nabla\bar{u}dx \\
&\le C\int_{\mathbb{R}^n}|u\nabla(-V)|^2 + |\nabla u|^2dx \\
&\le C\int_{\mathbb{R}^n}|u|^2dx + \int_{\mathbb{R}^n}|\nabla u|^2dx \\
&\le C||u_0||^2_{L^2} + CE_1(t).
\end{aligned}$$

This implies that

$$\begin{aligned}
E_1'(t) - CE_1(t) &\le C||u_0||^2_{L^2} \\
(E_1(t)e^{-Ct})'e^{ct} &\le C||u_0||^2_{L^2} \\
(E_1(t)e^{-Ct})' &\le e^{-ct}C||u_0||^2_{L^2} \\
E_1(T)e^{-CT} - E_1(0) &\le \int_0^T e^{-ct}C||u_0||^2_{L^2}dt \\
E_1(T) &\le e^{CT}\left(E_1(0) + C||u_0||^2_{L^2}\int_0^T e^{-ct}dt\right) \\
E_1(T) &\le C_T(E_1(0) + ||u_0||^2_{L^2}). \qquad (2.5.3)
\end{aligned}$$

This implies that for all $T > 0$ there exists a constant $C_T > 0$ such that

$$||u(t,\cdot)||^2_{H^1(\mathbb{R}^n)} \le C_T||u_0||^2_{H^1(\mathbb{R}^n)}.$$

In this situation we see that the $H^1$ norm of $u$ is bounded by the $H^1$ norm of $u_0$, but it is not exactly conserved.

We will now consider the Schrödinger equation on a manifold $M$ with metric $g$. Let $u$ be a solution to

$$\begin{cases} (D_t - \Delta_g)u = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{2.5.4}$$

and $u_0 \in \mathscr{S}(M)$. Note that $(-D_t - \Delta_g)\bar{u} = 0$. Let $E(t) = ||u(t,\cdot)||^2_{L^2(M)}$. Then,

$$\begin{aligned} E'(t) &= 2\operatorname{Re} i \int_M (D_t u)\bar{u} dM \\ &= 2\operatorname{Re} i \int_M (\Delta u)\bar{u} dM \\ &= -2\operatorname{Re} i \int_M (\nabla_g u) \cdot \nabla_g \bar{u} dx \\ &= -2\operatorname{Re} i \int_M |\nabla_g u|^2 dx \\ &= 0, \text{ since } |\nabla_g u|^2 \text{ is real.} \end{aligned}$$

Hence $E(t) = E(0)$ for all $t > 0$. Therefore, for all $t > 0$, we have $||u(t,\cdot)||^2_{L^2(M)} = ||u_0||^2_{L^2(M)}$.

Let $E_1(t) = ||\nabla_g u(t,\cdot)||^2_{L^2(M)}$. We use $\nabla_g$ to define $E_1$ because it commutes with $\Delta_g$, however $E + E_1$ is an equivalent norm to

$$||u(t,\cdot)||^2_{H^1(M)} = ||(1 + \partial_x + \partial_{\theta_+} + \partial_{\theta_-})u||^2_{L^2(M)}.$$

$$
\begin{aligned}
E_1'(t) &= \int_M (\nabla_g \partial_t u) \cdot \nabla_g \bar{u} dM + \int_M (\nabla_g u) \cdot \nabla_g \partial_t \bar{u} dM \\
&= -\int_M (\partial_t u) \Delta_g \bar{u} dM - \int_M (\Delta_g u) \partial_t \bar{u} dM \\
&= -\int_M \partial_t u D_t \bar{u} dM + \int_M D_t u \partial_t \bar{u} dM \\
&= -\frac{1}{i}\left( \int_M \partial_t u \partial_t \bar{u} dM - \int_M \partial_t u \partial_t \bar{u} dM \right) \\
&= 0
\end{aligned}
$$

Therefore $E_1(t) = E_1(0)$ for all $t > 0$. Hence, for all $t > 0$ we have $||u(t,\cdot)||^2_{H^1(M)} \leq ||u_0||^2_{H^1(M)}$. We see that on a manifold the $L^2$ and $H^1$ norms are conserved by using the properties of the Laplacian. In order to use the properties of the Laplacian for the $H^s$ norm we will define the norms in terms of powers of the Laplacian. In general, let $||u||_{\dot{H}^s} = ||(C - \Delta_g)^{s/2} u||_{L^2}$, where $(C - \Delta_g)^{s/2}$ is defined by the functional calculus presented in Section 2.4 for $0 < s < 1$. This norm is equivalent to any norm defining Sobolev spaces through pseudo-differential calculus. Note that

$$
(C - \Delta_g)^{s/2}(\Delta_g u) = \Delta_g((C - \Delta_g)^{s/2} u).
$$

Let $E_s(t) = ||u||^2_{\dot{H}^s}$. Then,

$$
\begin{aligned}
E_s'(t) &= \int_M ((C-\Delta_g)^{\frac{s}{2}} \partial_t u)(\Delta_g)^{\frac{s}{2}} \bar{u} dM + \int_M ((C-\Delta_g)^{\frac{s}{2}} u)(\Delta_g)^{\frac{s}{2}} \partial_t \bar{u} dM \\
&= \int_M ((C-\Delta_g)^{\frac{s}{2}} (\Delta_g u))(C-\Delta_g)^{\frac{s}{2}} \bar{u} dM - \int_M ((C-\Delta_g)^{\frac{s}{2}} u)(C-\Delta_g)^{\frac{s}{2}} (\Delta_g \bar{u}) dM \\
&= \int_M \Delta_g((C-\Delta_g)^{\frac{s}{2}} u)(C-\Delta_g)^{\frac{s}{2}} \bar{u} dM - \int_M (C-\Delta_g)^{\frac{s}{2}} u(\Delta_g (C-\Delta_g)^{\frac{s}{2}} \bar{u}) dM \\
&= -\int_M (\nabla_g (C-\Delta_g)^{\frac{s}{2}} u) \cdot (\nabla_g (C-\Delta_g)^{\frac{s}{2}} \bar{u}) dM + \int_M (\nabla_g (C-\Delta_g)^{\frac{s}{2}} u) \cdot (\nabla_g (C-\Delta_g)^{\frac{s}{2}} \bar{u}) dM \\
&= 0.
\end{aligned}
$$

Hence $E_s(t) = E_s(0)$ for all $t$.

# CHAPTER 3

# HISTORY OF LOCAL SMOOTHING RESULTS

## 3.1 General Local Smoothing Estimates

As stated in the introduction, local smoothing estimates have been studied by many over the years. The results by Doi [Doi96] and Craig-Kappeler-Strauss [CKS95] show that if a manifold is asymptotically Euclidean we gain 1/2 derivatives locally in $x$ averaged over time if and only if there are no trapped geodesic. Results by Marzoula-Metcalfe-Tataru [MMT08] and Ralston-Tau [RT07] extended these results to global in time local smoothing estimates on classes of asymptotically Euclidean manifolds. There have many been results to show how different kinds of trapping on asymptotically Euclidean manifolds affect the local smoothing estimates. In the following table we will summarize some of the results for local smoothing for asymptotically Euclidean manifolds.

| Trapping Type | Smoothing Estimates | References |
|---|---|---|
| None | 1/2 gain | [Doi96],[CKS95],[RT07],[MMT08] |
| Non-degenerate unstable | $1/2-\epsilon$ gain | [Dat09], [Bur04], [Chr08] |
| Stable | No Polynomial Gain | [Ral71] |
| Infinite Degenerate Unstable | No Polynomial Gain | [Chr18] |
| Finite Degenerate Unstable | Polynomial Gain | [CW13], [CM14], [CN22] |

From this we see that there is a range of a gain of 1/2 derivative with no trapping to a complete loss of polynomial gain in the degenerate unstable and stable cases. We will focus on the case in $\mathbb{R}^n$, which has no trapping, and the finite degenerate unstable cases where the polynomial gain depends on the nature of the unstable trapping.

## 3.2 Local Smoothing in $\mathbb{R}^2$

We have discussed how on $\mathbb{R}^n$ the local smoothing estimate is a gain of 1/2 derivative. We will prove this for $\mathbb{R}^2$. In the $\mathbb{R}^2$ case,

**Theorem 3.2.1.** *Suppose $u$ solves*

$$\begin{cases} (D_t - \Delta_{\mathbb{R}^2})u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

*where $u_0 \in H^{1/2}(\mathbb{R}^2)$. Then for every $T > 0$ there is a $C_T > 0$ such that*

$$\int_0^T ||\langle x\rangle^{-1}\partial_x u||^2 + ||\langle y\rangle^{-1}\partial_y u||^2 dt \leq C_T ||u_0||^2_{H^{1/2}}.$$

The important parts of the theorem is that the $\langle x\rangle^{-1}$ and $\langle y\rangle^{-1}$ terms localize in space and that the results implies a gain of 1/2 derivative locally averaged over time. (The $-1$ power is not optimal. Our focus in this thesis is the gain in derivatives, not the optimal localizing term.) We will go through the proof of the theorem because it illustrates one of the common techniques, a positive commutator argument, used to help prove local smoothing results in more complicated situations

*Proof.* Note that

$$[-\Delta, x\partial_x + y\partial_y] = -2\Delta.$$

We want to mimic this idea with a vector field $B = a(x)\partial_x + a(y)\partial_y$, so that $a(z) \approx z$ near 0 and is bounded as $|z| \to \infty$. Let $B = a(x)\partial_x + a(y)\partial_y$, where $a(z) = \arctan(z)$. Note that $a'(z)$ is non-negative, $a(z)$ and all derivatives of $a(z)$ are bounded, $a(z) \approx z$ near 0, and $a'(z) = \langle z\rangle^{-2}$.

**Remark 3.2.2.** *We could choose $a(z) = z\langle z\rangle^{-1}$ or any function such that $a'(z)$ is non-negative, $a(z)$ and all derivatives of $a(z)$ are bounded, $a(z) \approx z$ near $z = 0$. We chose* arctan$(z)$ *here because it provides a better localizing term than* $z\langle z\rangle^{-1}$ *and is already defined. In the proof of the main theorem we will define a function so that* $a(z) \equiv z$ *near* $z = 0$ *to make the calculations simpler.*

Now,

$$\begin{aligned}[B, -\Delta] =& [a(x)\partial_x + a(y)\partial_y, -\partial_x^2 - \partial_y^2] \\ &= [a(x)\partial_x, -\partial_x^2] + [a(y)\partial_y, -\partial_y^2] + [a(x)\partial_x, -\partial_y^2] + [a(y)\partial_y, -\partial_x^2] \\ &= (a''(x)\partial_x + 2a'(x)\partial_x^2) + (a''(y)\partial_y + 2a'(y)\partial_y^2). \end{aligned} \tag{3.2.3}$$

Since $u$ solves the Schrödinger equation,

$$\begin{aligned} 0 &= \int_0^T \langle B(D_t - \Delta)u, u\rangle dt \\ &= \int_0^T \langle B(D_t u), u\rangle + \langle B(-\Delta u), u\rangle dt \\ &= \int_0^T \langle Bu, D_t u\rangle + \langle B(-\Delta u), u\rangle dt - i\langle Bu, u\rangle|_0^T \\ &= \int_0^T \langle Bu, \Delta u\rangle + \langle B(-\Delta u), u\rangle dt - i\langle Bu, u\rangle|_0^T \\ &= \int_0^T \langle \Delta(Bu), u\rangle + \langle B(-\Delta u), u\rangle dt - i\langle Bu, u\rangle|_0^T \\ &= \int_0^T \langle [B, -\Delta]u, u\rangle dt - i\langle Bu, u\rangle\big|_0^T. \end{aligned}$$

Hence,

$$\int_0^T \langle [B, -\Delta]u, u\rangle dt = i\langle Bu, u\rangle\big|_0^T.$$

Using (3.2.3) gives

$$-\int_0^T \langle (a''(x)\partial_x + 2a'(x)\partial_x^2)u + (a''(y)\partial_y + 2a'(y)\partial_y^2)u, u\rangle dt = i\langle Bu, u\rangle\big|_0^T.$$

Moving the terms with single derivatives to one side gives

$$-\int_0^T \langle 2a'(x)\partial_x^2 u, u\rangle + \langle 2a'(y)\partial_y^2 u, u\rangle dt = i\langle Bu, u\rangle\Big|_0^T + \int_0^T \langle a''(x)\partial_x u, u\rangle dt + \int_0^T \langle a''(y)\partial_y u, u\rangle dt.$$

Integrating by parts in $x$ and $y$ and moving the terms where $\partial_x, \partial_y$ hit $a'(x)$ and $a'(y)$ respectively to the right hand side gives,

$$\int_0^T \langle 2a'(x)\partial_x u, \partial_x u\rangle + \langle 2a'(y)\partial_y u, \partial_y u\rangle dt$$
$$= i\langle Bu, u\rangle\Big|_0^T - \int_0^T \langle a''(x)\partial_x u, u\rangle dt - \int_0^T \langle a''(y)\partial_y u, u\rangle dt.$$

Substituting in for $a'(x)$ and $a'(y)$ gives

$$2\int_0^T ||\langle x\rangle^{-1}\partial_x u||_{L^2}^2 + ||\langle y\rangle^{-1}\partial_y u||_{L^2}^2 dt$$
$$= i\langle Bu, u\rangle\Big|_0^T - \int_0^T \langle a''(x)\partial_x u, u\rangle dt - \int_0^T \langle a''(y)\partial_y u, u\rangle dt.$$

Taking the absolute value of both sides gives,

$$2\int_0^T ||\langle x\rangle^{-1}\partial_x u||_{L^2}^2 + ||\langle y\rangle^{-1}\partial_y||_{L^2}^2 dt \leq \left|\langle a(x)\partial_x u, u\rangle\Big|_0^T\right| + \left|\langle a(y)\partial_y u, u\rangle\Big|_0^T\right|$$
$$+ \int_0^T |\langle a''(x)\partial_x u, u\rangle| dt + \int_0^T |\langle a''(y)\partial_y u, u\rangle| dt. \quad (3.2.4)$$

If we can show that the right hand side is bounded above by $C_T||u_0||_{H^{1/2}}^2$ for some constant $C_T$, then we have the desired bounded. This holds if $\left|\langle f(z)\partial_z u, u\rangle\right| \leq C||u_0||_{H^{1/2}}^2$ for $f(z) \in S^0$.

The four inner products satisfy this for $f(z) = a(z)$ or $f(z) = a''(z)$ and $z = x$ or $z = y$.

$$\begin{aligned}\Big|\langle f(z)\partial_z u, u\rangle\Big| &= \Big|\langle \partial_z u, f(z)u\rangle\Big| \\ &= \Big|\langle \langle D_z\rangle^{1/2}\langle D_z\rangle^{-1/2}\partial_z u, f(z)u\rangle\Big| \\ &= \Big|\langle \langle D_z\rangle^{-1/2}\partial_z u, \langle D_z\rangle^{1/2} f(z)u\rangle\Big|, \text{ by Lemma 2.3.10} \\ &\leq ||\langle D_z\rangle^{-1/2}\partial_z u||_{L^2} ||\langle D_z\rangle^{1/2} f(z)u||_{L^2}, \text{ by Cauchy-Schwarz.}\end{aligned}$$

Note that $\langle D_z\rangle^{-1/2}\partial_z = \mathrm{Op}^1(\langle \xi_z\rangle^{-1/2})\mathrm{Op}^1(\frac{1}{i}\xi_z)$. Since $\langle \xi_z\rangle^{-1/2} \in S^{-1/2}$ and $\frac{1}{i}\xi_z \in S^1$, Theorem 2.3.5 says $\langle D_z\rangle^{-1/2}\partial_z = \mathrm{Op}^1(a_1)$ for some $a_1 \in S^{1/2}$. Hence by Corollary 2.3.9 there is a constant $C_1$ such that

$$||\langle D_z\rangle^{-1/2}\partial_z u||_{L^2} \leq C_1||u||_{H^{1/2}}. \tag{3.2.5}$$

Note that $f(z) \in S^0$, so $\langle D_z\rangle^{1/2} f(z) = \mathrm{Op}^1(a_2)$ for some $a_2 \in S^{1/2}$. Hence by Corollary 2.3.9 there is a constant $C_2$ such that

$$||\langle D_z\rangle^{1/2}(f(z)\bar{u})||_{L^2} \leq C_2||u||_{H^{1/2}}. \tag{3.2.6}$$

Combining (3.2.5) and (3.2.6) gives

$$\Big|\langle f(z)\partial_z u, u\rangle\Big| \leq C_1||u||_{H^{1/2}} C_2||u||_{H^{1/2}} \leq C||u_0||^2_{H^{1/2}} \tag{3.2.7}$$

since $u$ solves the Schrödinger equation. Using (3.2.7) for all four inner products on the right hand side of (3.2.4) gives

$$\int_0^T ||\langle x\rangle^{-1}\partial_x u||^2_{L^2} + ||\langle y\rangle^{-1}\partial_y u||^2_{L^2} \leq C_T||u_0||^2_{H^{1/2}}$$

for some constant $C_T$.

□

This proof illustrates two of the important ideas throughout. The first is taking a vector field $B$ and commutating with $\Delta$ so that the terms we want to estimate are the same sign. We will use this technique to get the desired estimates away from trapped geodesics. Secondly, it illustrates that in Euclidean space we get a gain of 1/2 derivatives. This is the best local smoothing effect possible. On manifolds with trapping we get a worse smoothing effect.

### 3.3 Local Smoothing on a Surface of Revolution with Degenerate Trapping

In this section we are going to consider a surface of revolution with degenerate trapping. From the previous section we saw that we could use a positive commutator argument to get a local smoothing estimate for the Schrödinger equation on $\mathbb{R}^2$. This same idea will be used to get a local smoothing estimate away from the trapped geodesic, allowing us to localize near the trapped geodesic. Since the manifold is a surface of revolution, we can separate variables to reduce a 2-dimensional problem down to 1 dimension. From there we will use a $TT^*$ argument to show that a microlocal bound on a resolvent estimate gives the desired result. Finally, we will use a scaling argument and another positive commutator argument to prove the resolvent estimate.

#### 3.3.1 Setup

We are going to consider the Schrödinger equation on a surface of revolution. Let $M$ be the manifold $\mathbb{R}_x \times \mathbb{S}^1$ with the metric $dx^2 + A^2(x)d\theta^2$ where $A(x) = (1 + x^{2m})^{\frac{1}{2m}}$ and $m$ is an integer greater than or equal to 2. This surface of revolution will look similar to Figure 3.3.1.

Notice that there is only a trapped geodesic at $x = 0$, which circles the surface of revolution and that the metric is asymptotically Euclidean. We say this is a degenerate trapped geodesic of order $2m$ since $(A^2)^{(k)} = 0$ for all positive integers $k < 2m$. Recall that $A'$ determines how the $x$ value of a geodesic curve changes and $A''/A$ determines the curvature for surfaces of revolution. If more derivatives of $A^2$ are 0 at the critical point, then we should expect the

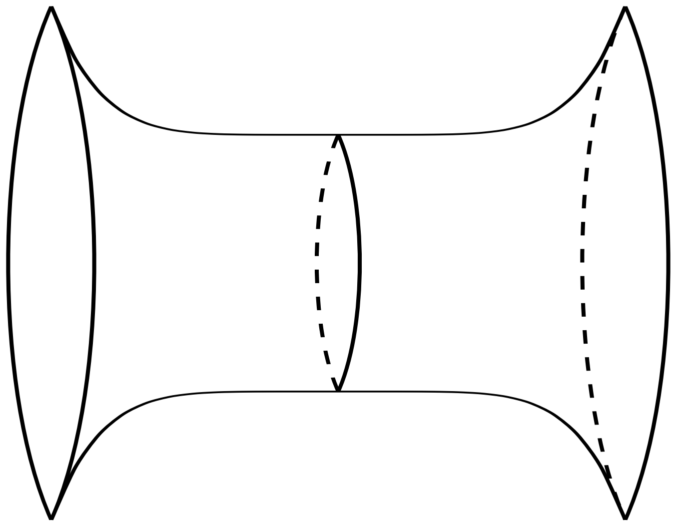

Figure 3.1: Surface of Revolution with Degenerate Trapping

surface to look more cylindrical near the critical point. This leads us to believe that the degeneracy of the trapping will affect the local smoothing estimates and that the order is important.

The local smoothing result proven in [CW13] is the following,

**Theorem 3.3.1** (Christianson-Wunsch ('13)). *Suppose $M$ is the manifold $\mathbb{R}_x \times \mathbb{R}_\theta/2\pi\mathbb{Z}$ with the metric $dx^2 + A^2(x)d\theta^2$ where $A(x) = (1 + x^{2m})^{1/2m}$ with $m \in \mathbb{Z}$ and $m \geq 2$. Suppose $u(t, x, \theta)$ is a solution to the Schrödinger equation on $M$ with initial condition $u(0, x, \theta) = u_0(x, \theta)$ where $u_0 \in H^s$ for $s \geq \frac{m}{m+1}$. Then for all $T > 0$ there is a $C_T > 0$ such that*

$$\int_0^T ||\langle x\rangle^{-3/2}u||_{H^1}^2 dt \leq C_T(||\langle D_\theta\rangle^{m/(m+1)}u_0||_{L^2}^2 + ||\langle D_x\rangle^{1/2}u_0||_{L^2}^2).$$

The paper by Christianson and Wunsch also shows that the best possible result is a gain of $1/(m+1)$ derivatives locally averaged over time in the $\theta$-direction and a gain in $1/2$ derivatives locally averaged over time in the $x$-direction.

**Remark 3.3.2.** *In the $m = 1$ case in the formula we would get a $1/2$ gain, however there is only a $1/2 - \epsilon$ gain in derivatives instead of $1/2$, since this is the non-degenerate unstable* *trapping case. The same proof will work to get a log squared loss if we adjust the commutant a in* (3.3.14)*. As $m \to \infty$, the degenerate trapping should become worse, which we see that it approaches no polynomial gain in derivatives. This situation as $m \to \infty$ is similar to the degenerate unstable trapping case studied in* [Chr18].

### 3.3.2 Proof Overview

We will follow the proof directly from [CW13] to go over the strategies used. To start recall that,

$$\Delta f = (\partial_x^2 + A^{-2}\partial_\theta^2 + A^{-1}A'\partial_x)f.$$

After a conjugation argument to get rid of the first order derivatives terms we can instead study the operator,

$$\tilde{\Delta} f = (\partial_x^2 + A^{-2}\partial_\theta^2 - V_1(x))f$$

where

$$V_1(x) = \frac{1}{2}A''A^{-1} - \frac{1}{4}(A')^2A^{-2}.$$

Initially we use a positive commutator argument. This technique is similar to the proof of local smoothing in the Euclidean case. This positive commutator argument will give

$$\int_0^T (||\langle x\rangle^{-1}\partial_x u||_{L^2}^2 + ||\,|x|^m\langle x\rangle^{-m-3/2}\partial_\theta u||_{L^2}^2)dt \leq C||u_0||_{H^{1/2}}^2.$$

Note that this results shows that there is perfect local smoothing in the $x$-direction and that away from $x = 0$ there is perfect local smoothing in the $\theta$-direction. Now, we need to show what happens near $x = 0$.

We can now separate variables to get that

$$u(t, x, \theta) = \sum_k e^{ik\theta}u_k(t, x)$$

with initial condition

$$u_0(x,\theta) = \sum_k e^{ik\theta} u_{0,k}(x).$$

After separation of variables and local smoothing away from the trapping we get the desired theorem if we can show, for $|k| \geq 1$,

$$\int_0^T ||\chi(x) k u_k||_{L^2}^2 dt \leq C(||\langle k\rangle^{m/(m+1)} u_{0,k}||_{L^2}^2 + ||u_{0,k}||_{H^{1/2}}^2)$$

for some $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\chi(x) \equiv 1$ near $x = 0$. This comes from the fact that $\partial_\theta u(t,x,\theta) = \sum_k k e^{ik\theta} u_k(t,x)$, so a derivatives in $\theta$ correspond with multiplying by $k$. Since we have perfect local smoothing in the $x$-direction we only need to show upper bounds on the $\theta$-direction near $x = 0$.

The next technique we will use is to break up $u$ into parts where the angular frequency is high compared to the radial frequency and where the angular frequency is low compared to the radial frequency. We will call these the high and low frequency parts respectively.

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ be an even function with $\psi(r) \equiv 1$ for $|r| \leq 1$ and $\psi(r) \equiv 0$ for $|r| \geq 2$. Let

$$u = u_{\text{hi}} + u_{\text{lo}}$$

where

$$u_{\text{hi}} = \psi(D_x/k)u, \; u_{\text{lo}} = (1 - \psi(D_x/k))u.$$

Intuitively, the obstruction to a locally smoothing estimate is the trapped geodesic which goes around the surface. If the radial frequency is high, then the waves will spread out quicker. The issue is when the angular frequency is high and the waves stay circling near the trapped geodesic at $x = 0$ for a long time. Specifically, on $\text{supp}\,(1-\psi)$, we get $|k| \lesssim \langle D_x \rangle$, so we can control $u_{lo}$ by the radial derivative where we have local smoothing estimates up to a compactly supported region similar to $\psi$. Taking $\tilde{\psi}$ to have the same properties as $\psi$ but is 1 on supp $(\psi)$, this breakdown allows us to get the desired result if we can show

$$\int_0^T ||\chi k\tilde{\psi}(D_x/k)u||_{L^2}^2 dt \leq C||k^{m/(m+1)}u_0||_{L^2}^2. \tag{3.3.3}$$

To handle the high frequency argument we will do a $TT^*$ argument. Although in our case it will be $FF^*$, since we used $T$ already. Let

$$P_k = D_x^2 + A^{-2}(x)k^2 + V_1(x).$$

and let $F(t)$ be defined by

$$F(t)g = \chi(x)\psi(D_x/k)k^{\frac{1}{m+1}}e^{-itP_k}g$$

where $e^{-itP_k}$ is the Schrödinger propagator. If we can show $F$ is a bounded mapping $F : L_x^2 \to L_t^2([0,T])L_x^2$, then

$$||\chi k\tilde{\psi}(D_x/k)u||_{L_t^2([0,T]);L_x^2} = ||k^{\frac{m}{m+1}}F(t)u_0||_{L_t^2([0,T]);L_x^2} \leq C||k^{\frac{m}{m+1}}u_0||_{L_x^2}$$

holds for some constant $C$ and gives the desired estimate in (3.3.3). There is a bounded mapping $F : L_x^2 \to L_t^2([0,T])L_x^2$ if and only if there is a bounded mapping $FF^* : L_t^2([0,T])L_x^2 \to$

$L_t^2([0,T])L_x^2$ where $F^*$ is the adjoint of $F$.

$$\begin{aligned}
\langle F(t)g(x), q(t,x)\rangle_{L_t^2([0,T])L_x^2} &= \int_0^T \int_{\mathbb{R}} F(s)g(x)\overline{q(s,x)}dxds && (3.3.4)\\
&= \int_0^T \int_{\mathbb{R}} \chi(x)\psi(D_x/k)k^{\frac{1}{m+1}}e^{-isP_k}g(x)\overline{q(s,x)}dxds && (3.3.5)\\
&= \int_0^T \int_{\mathbb{R}} \psi(D_x/k)k^{\frac{1}{m+1}}e^{-isP_k}\chi(x)g(x)\overline{q(s,x)}dxds && (3.3.6)\\
&= \int_0^T \int_{\mathbb{R}} k^{\frac{1}{m+1}}e^{-isP_k}\psi(D_x/k)\chi(x)g(x)\overline{q(s,x)}dxds && (3.3.7)\\
&= \int_0^T \int_{\mathbb{R}} g(x)\overline{k^{\frac{1}{m+1}}e^{isP_k}\psi(D_x/k)\chi(x)q(s,x)}dxds && (3.3.8)\\
&= \int_{\mathbb{R}} g(x)\int_0^T \overline{k^{\frac{1}{m+1}}e^{isP_k}\psi(D_x/k)\chi(x)q(s,x)}dsdx && (3.3.9)\\
&= \langle g(x), (F^*q)(x)\rangle_{L_x^2} && (3.3.10)
\end{aligned}$$

where

$$(F^*q)(x) = \int_0^T k^{\frac{1}{m+1}}e^{isP_k}\psi(D_x/k)\chi(x)q(s,x)ds.$$

Line (3.3.6) follows since we are just mulplying by $\chi(x)$. Line (3.3.7) follows from $\psi(D_x/k)$ being self-adjoint. Line (3.3.8) comes from $e^{isP_k}$ being unitary.

Applying $FF^*$ to a function $f$ gives

$$FF^*f(t,x) = \chi(x)\psi(D_x/k)k^{\frac{2}{m+1}}\int_0^T e^{i(t-s)P_k}\psi(D_x/k)\chi(x)f(s,x)ds.$$

We want to break $FF^*f(t,x)$ up into two pieces $FF^*f(t,x) = \chi\psi(v_1+v_2)$ so that $v_j$ solve a differential equation. Then we can apply a resolvent estimate to show $FF^*$ is a bounded operator. If we let

$$v_1 = k^{\frac{2}{m+1}}\int_0^t e^{i(t-s)P_k}\psi(D_x/k)\chi(x)f(s,x)ds$$

and

$$v_2 = k^{\frac{2}{m+1}}\int_t^T e^{i(t-s)P_k}\psi(D_x/k)\chi(x)f(s,x)ds,$$

then $v_j$ solves the equation

$$(D_t + P_k)v_j = \pm i k^{\frac{2}{m+1}} \psi \chi f.$$

Now we take the Fourier transform with respect to $t$ to get

$$(\tau + P_k)\hat{v_j} = \pm i k^{\frac{2}{m+1}} \psi \chi \hat{f}.$$

The idea is if a resolvent exists, then we have

$$\chi\psi\hat{v_j} = \pm i k^{\frac{2}{m+1}} \chi\psi(\tau \pm i0 + P_k)^{-1}\psi\chi\hat{f}. \tag{3.3.11}$$

and

$$||\chi\psi\hat{v_j}||_{L_t^2([0,T])L_x^2} \leq C||\hat{f}||_{L_t^2([0,T])L_x^2} \tag{3.3.12}$$

for some constant $C$. Since the Fourier transform is unitary, given (3.3.12) we get

$$||FF^*f||_{L_t^2([0,T])L_x^2} \leq ||\chi\psi v_1||_{L_t^2([0,T])L_x^2} + ||\chi\psi v_2||_{L_t^2([0,T])L_x^2} \leq 2C||f||_{L_t^2([0,T])L_x^2}.$$

Take $-z = \tau k^{-2}$ and $h = k^{-1}$, then showing (3.3.11) and (3.3.12) is given by proving

$$||\chi\psi(-zh^{-2} \pm i0 + D_x^2 + A^{-2}(x)h^{-2} + V_1(x))^{-1}\psi\chi||_{L_x^2 \to L_x^2} \leq Ch^{\frac{2}{m+1}}.$$

After factoring out $h^2$ on the left hand side we get the necessary estimate to show will be

$$||\chi(x)\psi(hD_x)(-z \pm i0 + (hD_x)^2 + V)^{-1}\psi(hD_x)\chi(x)||_{L^2 \to L^2} \leq Ch^{-2(\frac{m}{m+1})} \tag{3.3.13}$$

where $V = A^{-2}(x) + h^2 V_1(x)$.

This allows us to reduce the local smoothing estimate to proving this microlocal resolvent estimate. This low and high frequency decomposition and $FF^*$ argument will also be used when we get to the multi-warped case. However, we will have two parameters $p, n$ instead of just $k$. This means we will have to be more careful with what we mean by the low and high frequency parts. We will also need to isolate at a critical point that is moving depending on $p, n$. However, the ideas used in the degenerate surface of revolution case remain the same.

Now, showing this resolvent estimate is equivalent to

$$C||(Q_1 - z)v||_{L^2(x)} \geq h^{2m/(m+1)}||v||_{L^2(x)}$$

for $Q_1 = (hD_x)^2 + V - h^2V_1$ and $v = \varphi^w u$ with $\varphi \in S^0(\mathbb{R})$ with compact support in $\{|(x, \xi)| \leq \epsilon\}$.

To do this we notice that

$$C||(Q_1 - z)v||_{L^2(x)} \, ||v||_{L^2(x)} \geq |\langle [Q_1 - z, a^w]v, v\rangle| \tag{3.3.14}$$

when $a^w$ is a bounded operator and $Q_1$ is essentially self-adjoint.

We will go over the intuition behind the choice of commutant $a$, review why we need to use a two parameter calculus in this proof and where we derive the powers of $h$ that we use to scale $x$ and $\xi$. We introduce the commutator in (3.3.14) so that due to the Weyl quantization the first term is given by $h\langle \operatorname{Op}_h^w(H_{q_1}(a))v, v\rangle$ where $q_1$ is the symbol of $Q_1$ and we only have odd order terms. The main idea is that $q_1 \sim \xi^2 - x^{2m}$, which has a negative term, however we can get rid of the negative term using the correct commutant. If $a = h^{-1}x\xi$, then

$$h\{q_1, a\} = \{\xi^2 - x^{2m}, x\xi\} = 2\xi^2 + 2mx^{2m}.$$

However, the issue is that $a$ is unbounded, but we need $a \sim h^{-1}x\xi$ near 0. In [CW13] the authors define a function $\Lambda$ so that $\Lambda(s) \sim s$ near $s = 0$ and bounded. Then, $a \sim h^{-1}\Lambda(x)\Lambda(\xi)$ will almost give the proper $a$, however this is still unbounded in terms of $h$. To solve this we will take $a \sim \Lambda(h^{-\alpha}x)\Lambda(h^{-\beta}\xi)$ with $\alpha + \beta = 1$, however $hH_{q_1}(a) \approx (\xi^2 + x^{2m})$ will only be true on a $h$ dependent region. To solve this we will rescale $x = h^\alpha X$ and $\xi = h^\beta \Xi$.

Then, $hH_{q_1}(a) \approx h(\xi\Xi h^{-\alpha} + x^{2m-1}Xh^{-\beta}) = h(\Xi^2 h^{-\alpha+\beta} + X^{2m}h^{-\beta+(2m-1)\alpha})$. We want $h^{-\alpha+\beta} = h^{-\beta+(2m-1)\alpha}$ and $\beta + \alpha = 1$ to get the lowest power of $h$. Solving the two equations gives $\alpha = 1/(m+1)$ and $\beta = m/(m+1)$. Hence,

$$hH_{q_1}(a) \approx h^{2m/(m+1)}(X^{2m} + \Xi^2)$$

for $X, \Xi \leq C$ for some constant independent of $h$.

**Remark 3.3.15.** *The case where $x = h^\alpha X$ and $\xi = h^\beta \Xi$ for $\alpha, \beta = 1/2$ is considered the standard rescaling. However, since $\alpha + \beta = 1$ we will still have*

$$\langle h\mathrm{Op}_h^w(H_{q_1}(a))u, u\rangle \approx h^{2m/(m+1)}\langle \mathrm{Op}_1^w(X^{2m} + \Xi^2)u, u\rangle.$$

The last issue is that we will have higher order terms. Specifically looking at the third order term we will get a term similar to $h^3h^{-3\beta}h^{(2m-3)\alpha}X^{2m-2} = h^{2m/(m+1)}X^{2m-2}$. There is no gain in powers of $h$ to absorb the higher order terms which is an issue since $X^{2m-2} > X^{2m}$ for $X < 1$. This is why we need to introduce a second parameter $\tilde{h}$. In [CW13], they use the scaling $x = (h/\tilde{h})^\alpha X$ and $\xi = (h/\tilde{h})^\beta \Xi$. With this scaling the third order and higher terms will now have the same powers of $h$, but gain powers of $\tilde{h}$. By taking $\tilde{h}$ sufficiently small we can absorb the higher order terms.

**Remark 3.3.16.** *We have glanced over the details here to explain the intuition for the argument and the need for the two parameter calculus. Calculating $[Q_1 - z, a^w]$ will be the major issue for proving the local smoothing estimate for the multi-warped product with two infinite ends case and will be covered in full detail in Chapter 4.*

### 3.3.3 Conclusion

The main techniques we will use in the multi-warped product case are those used in [CW13]. The difference is that we will have a range of trapping rather than a single point. We will then separate variables to reduce it to a 1-dimensional problem. This is the advantage to studying a multi-warped product. We will first prove local smoothing away from the trapping. We will then break down into the high and low angular frequency parts. The difference in the multi-warped product case is that we will have two angular frequencies. Then, we will do a $TT^*$ argument to reduce the local smoothing estimate to a microlocal resolvent estimate. Finally, we will do a similar commutator and two parameter scaling argument. The main difference here is that $h$ will be dependent on both angular frequencies and estimating the terms of commutator is more complicated.

### 3.4 Other Related Results

Next we will cover two other related results to the paper [CW13]. These proofs will follow a similar process, however there will be changes that are necessary.

#### 3.4.1 Inflection Trapping

In the paper [CM14] by Christianson and Metcalfe, they extend the degenerate trapping on a surface of revolution to inflection-transmission trapping. In this case we will have the following result,

**Theorem 3.4.1** (Christianson-Metcalfe ('14)). *Suppose $M$ is the manifold $\mathbb{R}_x \times \mathbb{S}^1$ with the metric $dx^2 + A^2(x)d\theta^2$ where*

$$A^2(x) = 1 + \int_0^x y^{2m_1-1}(y-1)^{2m_2}/(1+y^2)^{m_1+m_2-1}dy$$

*for positive integers $m_1$ and $m_2$. Suppose $u$ solves*

$$\begin{cases} (D_t - \Delta_M)u(t,x) = 0 \\ u|_{t=0} = u_0 \in H^s \end{cases}$$

*for some $s > 0$ sufficiently large. Then, for all $\infty > T > 0$ there is a $C_T > 0$ such that*

$$\int_0^T ||\langle x\rangle^{-1}\partial_x u||_{L^2}^2 + ||\langle x\rangle^{-3/2}\partial_\theta u||_{L^2}^2 dt$$
$$\leq C_T(||\langle D_\theta\rangle^{\beta(m_1,m_2)}u_0||_{L^2}^2 + ||\langle D_x\rangle^{1/2}u_0||_{L^2}^2)$$

*where*

$$\beta(m_1, m_2) = \max\left(\frac{m_1}{m_1+1}, \frac{2m_2+1}{2m_2+3}\right).$$

Figure 3.2 from [CM14] shows roughly what the manifold $M$ looks like.

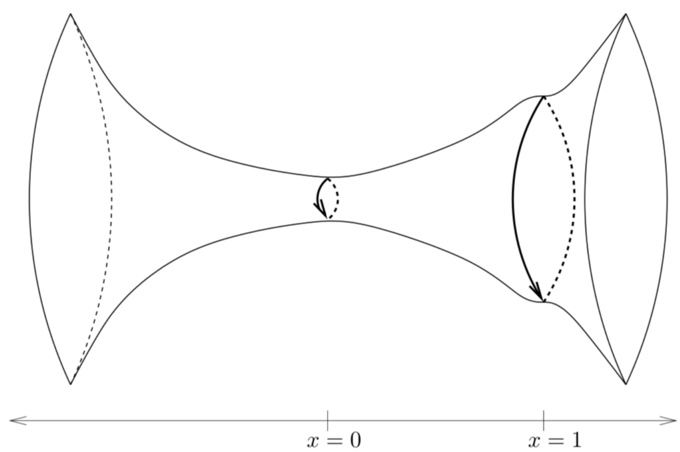


Figure 3.2: Surface of revolution with inflection point trapping from [CM14]

There is degenerate trapping of order $2m_1$ at $x = 0$ as in [CW13] and inflection-transmission trapping at $x = 1$ of order $2m_2 + 1$. This proof follows the same strategy as [CW13]. The main difference is that we have two points of trapping to deal with. This requires a positive commutator argument to get local smoothing away from $x = 0$ and $x = 1$. Then, we do the same separation of variables, high and low frequency decomposition and $TT^*$ argument. The difference is that we then have to isolate around the two critical points instead of one. The estimate around $x = 0$ follows from [CW13], while the estimate around $x = 1$ is proved using a scaling argument and two-parameter calculus again. This estimate is different because $A'$ does not change sign at $x = 1$. The main change in this paper is showing that we can isolate around each trapped geodesic and then showing the necessary estimates in the inflection-transmission trapping case.

### 3.4.2 Multi-Warped Inflection Trapping

In [CN22], Christianson and the author extend the inflection point trapping estimate to multi-warped products. In this case we consider the functions $A_1^2$ and $A_2^2$ in Figure 3.3.

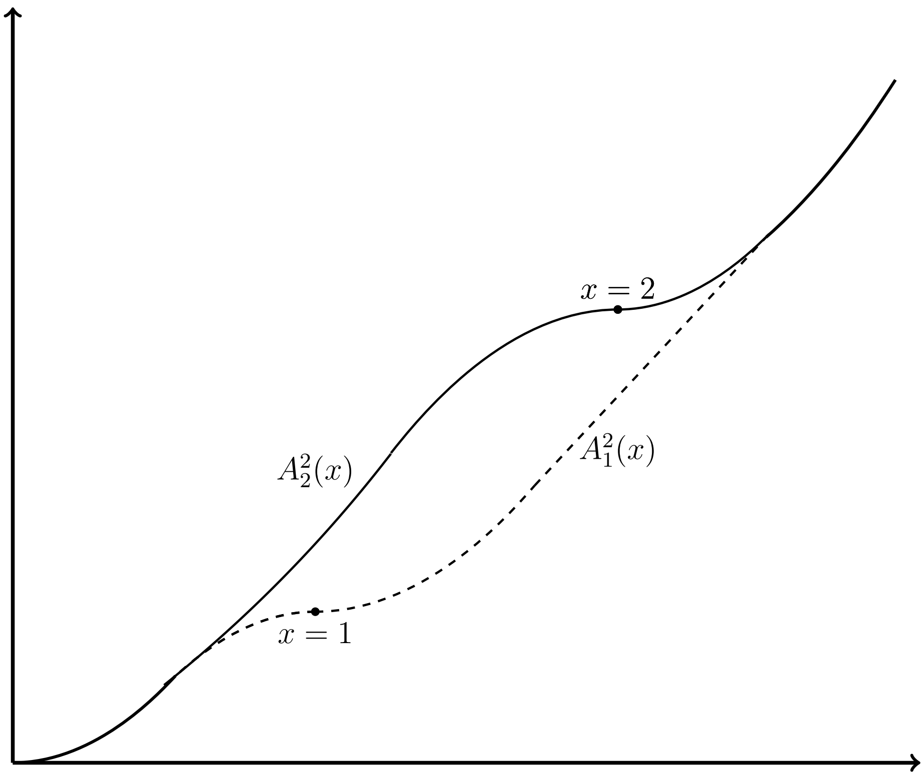


Figure 3.3: The functions $A_1^2$ and $A_2^2$.

Then, we defined the multi-warped product $X = \mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{S}^1$ with the metric

$$g = dx^2 + A_1^2(x)d\theta^2 + A_2^2(x)d\omega^2.$$

Notice that there is inflection-transmission trapping at $x = 1$ of order $2m_1 + 1$ and $x = 2$ of order $2m_2 + 1$, but the trapping at $x = 1$ is only in the $\theta$-direction and at $x = 2$ the trapping is only in the $\omega$-direction.

The result we get is,

**Theorem 3.4.2.** *Let $(M, g)$ be the multi-warped product constructed above. Suppose $u$ solves the Schrödinger equation on $M$ with initial condition $u_0 \in \mathscr{S}(M)$. Let $m = \max(m_1, m_2)$. Then for each $T > 0$ there exists a constant $C$ such that*

$$\int_0^T \| \langle x \rangle^{-3/2} u\|^2_{H^1(M)} dt \leq C\|u_0\|^2_{H^{\frac{2m+1}{2m+3}}(M)}. \tag{3.4.3}$$

This proof follows the same strategy as [CW13]. The main difference is that we have two angular directions, however the trapping is only in one direction at each point. This allows us to separate variables in only one direction to get the estimates at $x = 1$ and $x = 2$ separately. Once we separate variables, we have a function with two parameters rather than one parameter like in [CW13] and [CM14]. We solve this issue by showing that the extra dimension does not harm the local smoothing estimate, because we have local smoothing in the extra direction near $x = 1$. After making sure this is the case and working through the low frequency estimates this case reduces to the result proven in [CM14].

**Remark 3.4.4.** *The result proven in this paper will follow a similar process to* [CW13], [CM14], *and* [CN22]. *One main difference is that we have trapping that when projected to the $x$-direction after separating variables forms a countable dense subset of an interval. Additionally, this trapping is dependent on both angular frequencies. This prevents a straightforward positive commutator argument. The other main difference is that we will actually separate variables in both angular directions, since the trapping is not isolated in a single direction.*

# CHAPTER 4

# MULTI-WARPED PRODUCTS WITH TWO INFINITE ENDS

## 4.1 Introduction

In this chapter we will use the proof methods used in [CW13], [CM14], and [CN22] to show a local smoothing result of the Schrödinger equation on a multi-warped product where the projection of the trapped trajectories onto the $x$-coordinate after separating variables is a countable dense subset of the interval $[-\varepsilon, \varepsilon]$. We will start by considering the multi-warped manifold $M = \mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$ where the metric is given by

$$g(x, \theta, \omega) = dx^2 + U_-(x)^2 d\theta_-^2 + U_+(x)^2 d\theta_+^2.$$

The $U_-^2$ and $U_+^2$ will look roughly like the following functions in Figure 4.1 near 0.

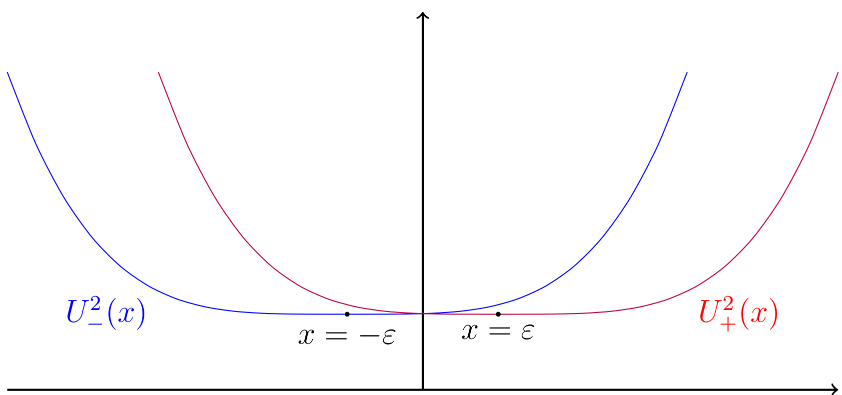


Figure 4.1: Warping Functions

Note that the Laplacian will be determined by $U_\pm^{-1}$, so that there is finite degenerate unstable trapping in the $\theta_-$ direction at $-\varepsilon$ and there is finite degenerate unstable trapping in the $\theta_+$ direction at $\varepsilon$. We will want $g$ to be Euclidean outside of a compact set.

The important fact to notice from this setup compared to the setup in [CN22] is that for $x \in (-\varepsilon, \varepsilon)$, $U_-^2(x)$ is increasing while $U_+^2(x)$ is decreasing. This causes trapping on this region rather than being isolated at two points in the radial direction like in [CN22]. As a result, we will get perfect local smoothing only outside of $[-\varepsilon, \varepsilon]$ and will gain less than 1/2 derivatives on the region depending on the point $x \in [-\varepsilon, \varepsilon]$ and the nature of the trapping.

### 4.2 Set-up

The picture given in Figure 4.1 will guide the setup of the problem. We will make slight changes to help with calculations.

To start we will define the following functions to construct the metric. Fix $1 > \varepsilon > 0$ First, let $K \in \mathcal{C}_c^\infty(\mathbb{R})$ be a cutoff function so that

$$K(x) = \begin{cases} 1 & -2\varepsilon \leq x \leq 2\varepsilon \\ 0 & 4\varepsilon \leq |x| \end{cases}$$

$K(x) \geq 0$, $K'(x) \geq 0$ on $[-4\varepsilon, -2\varepsilon]$ and $K'(x) \leq 0$ on $[2\varepsilon, 4\varepsilon]$, $K(x)$ is even and has a smooth square root $K^{1/2}(x)$. Let $m_1, m_2 \geq 2$ be a positive integer. Define the function

$$V_+(x) = K(x)(\tilde{M} - (x-\varepsilon)^{2m_1}) + (1 - K(x))x^{-2}$$

and

$$V_-(x) = K(x)(\tilde{M} - (x+\varepsilon)^{2m_2}) + (1 - K(x))x^{-2}$$

where $\varepsilon > 0$ and $\tilde{M} > (5\varepsilon)^{2m_1}$ and $\tilde{M} > (5\varepsilon)^{2m_2}$. The choice of sign is so that $V_+$ has a critical point at $x = \varepsilon$ and $V_-$ has a critical point at $x = -\varepsilon$. The choice of $\tilde{M}$ is so that $K(x)(\tilde{M} - (x-\varepsilon)^{2m_1}) > 0$ and $K(x)(\tilde{M} - (x+\varepsilon)^{2m_2}) > 0$. The nature of $V_+$ is such that it has a single degenerate critical point of order $2m_1$ at $x = \varepsilon$ and is equal to $x^{-2}$ outside of a compact set. The nature of $V_-$ is such that it has a single degenerate critical point of order

$2m_2$ at $x = -\varepsilon$ and is equal to $x^{-2}$ outside of a compact set.

Let $U_\pm = V_\pm^{-1/2}$ for convenience. With these definitions we consider a manifold $M = \mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$ with coordinates $(x, \theta_+, \theta_-)$ and the metric

$$g = dx^2 + V_-^{-1}(x)d\theta_-^2 + V_+^{-1}(x)d\theta_+^2 = dx^2 + U_-^2(x)d\theta_-^2 + U_+^2(x)d\theta_+^2.$$

Then,

$$\Delta_M = \partial_x^2 + V_-(x)\partial_{\theta_-}^2 + V_+(x)\partial_{\theta_+}^2 + \frac{U_+'U_- + U_+U_-'}{U_+U_-}\partial_x.$$

We will prove the following theorem,

**Theorem 4.2.1** (Main Result)**.** *Let $M$ be the multi-warped product with $\Delta_M$ constructed above with the case $m_1 = m_2 = 2$. Let $u$ be a solution to the Schrödinger equation on $M$ with initial condition $u_0 \in \mathscr{S}(M)$. For each $T > 0$ there exists a constant $C$ such that*

$$\int_0^T ||\langle x\rangle^{-3/2}u||_{H^1(M)}^2 dt \leq C||u_0||_{H^{2/3}(M)}^2.$$

Additionally, as a result of the proof of this theorem we have the following resolvent estimate,

**Theorem 4.2.2.** *Let $R(\lambda) = (-\Delta_M - \lambda^2)^{-1}$ denote the resolvent on $M$. For any $\chi \in \mathcal{C}_c^\infty(M)$, there exists a constant $C > 0$ such that for $\lambda \in \mathbb{R}$ and $\lambda \gg 0$,*

$$||\chi R(\lambda - i0)\chi||_{L^2 \to L^2} \leq C\lambda^{-2/3}.$$

**Remark 4.2.3.** *The local smoothing away from the trapping in Proposition 4.3.42 holds for all integer $m_1, m_2 \geq 2$. The estimate in* (4.4.13) *to prove local smoothing near the trapping is where we need to reduce to the case $m_1 = m_2 = 2$.*

Before we begin, we want to do a conjugation argument to reduce the Laplacian and volume form $U_+U_-$. Let $T = (U_+U_-)^{1/2}$. Then,

$$\tilde{\Delta} = T\Delta T^{-1} = \partial_x^2 + V_-(x)\partial_{\theta_-}^2 + V_+(x)\partial_{\theta_+}^2 + V_1(x)$$

for

$$V_1(x) = \frac{U_+^2(U_-')^2 + (U_+')^2U_-^2 - 2U_+^2U_-''U_- - 2U_+''U_+U_-^2 - 2U_+U_-U_+'U_-'}{4U_+^2U_-^2}.$$

Let $\tilde{u} = Tu$ and $\tilde{u}_0 = Tu_0$. As explained in Section A.3, if $\tilde{u}$ solves

$$\begin{cases} (D_t - \tilde{\Delta})\tilde{u}(t,x) = 0 \\ \tilde{u}(0,x) = \tilde{u}_0(x), \end{cases} \tag{4.2.4}$$

then $u$ is a solution to

$$\begin{cases} (D_t - \Delta)u(t,x) = 0 \\ u(0,x) = u_0(x). \end{cases} \tag{4.2.5}$$

Additionally, if

$$\int_0^T ||\langle x\rangle^{-3/2}\tilde{u}||^2_{H^1(X,dxd\theta_-d\theta_+)}dt \le C||\tilde{u}_0||^2_{H^{2/3}(X,dxd\theta_-d\theta_+)}$$

for some constant $C > 0$ holds, then

$$\int_0^T ||\langle x\rangle^{-3/2}u||^2_{H^1(X,dVol)}dt \le C'||u_0||^2_{H^{2/3}(X,dVol)}$$

holds as well for some constant $C' > 0$.

## 4.3 Local smoothing away from trapping

### 4.3.1 Separation of variables

In [CN22], we would now do a positive commutator to get local smoothing away from the trapping. The idea is that we need to use a function $f(x)$ such that $-f(x)V_-'(x) \geq 0$ and $-f(x)V_+'(x) \geq 0$ for all $x$. However, this argument would only work outside of the region $[-\varepsilon, \varepsilon]$, since $V_+'$ and $V_-'$ have different signs on this range. To handle this interval we will separate variables first. Due to the manifold being a multi-warped product, we can write a solution $u$ to $(D_t - \tilde{\Delta}_M)u = 0$ with $u(0, x) = u_0(x) \in \mathscr{S}$ as

$$u(t, x, \theta_+, \theta_-) = \sum_{p,n} u_{p,n}(t, x) e^{ip\theta_+} e^{in\theta_-}$$

and

$$u_0(x, \theta_+, \theta_-) = \sum_{p,n} u_{p,n,0}(x) e^{ip\theta_+} e^{in\theta_-}$$

where

$$(D_t - \tilde{\Delta}_{p,n}) u_{p,n}(t, x) = (D_t + D_x^2 + p^2 V_+(x) + n^2 V_-(x) + V_1(x)) u_{p,n}(t, x) = 0$$

and

$$u_{p,n}(0, x) = u_{p,n,0}(x).$$

We can see that $u_{p,n}$ is the solution to a 1-dimensional Schrödinger equation. Let

$$V_{p,n} = p^2 V_+(x) + n^2 V_-(x). \tag{4.3.1}$$

### 4.3.2 Norm conservation

Before we compute the positive commutator argument we will show that the $H^s$ norm is conserved for each $u_{p,n}$. For each $u_{p,n}(t,x)$ we define

$$l^s_{p,n}(x,\xi) = (\xi^2 + (C + p^2V_+(x) + n^2V_-(x) + V_1(x)))^s$$

where $C > 0$ is chosen so that $(C + p^2V_+(x) + n^2V_-(x) + V_1(x)) \geq 0$. This guarantees that $l^s_{p,n}$ is real. Let $b_{p,n}(x,\xi) = \xi^2 + p^2V_+(x) + n^2V_-(x) + V_1(x)$ so that $\tilde{\Delta}_{p,n} = b^w_{p,n}$.

$$[\mathrm{Op}^w(l^s_{p,n}), \tilde{\Delta}_{p,n}] = 2\left(\sum_{k=0}^{\infty} \frac{(iA(D))^{2k+1}}{(2k+1)!}(l^s_{p,n}(x,\xi)b_{p,n}(y,\eta))\Big|_{x=y,\xi=\eta}\right)^w \tag{4.3.2}$$

where the right hand side of (4.3.2) is a formal asymptotic expansion. First note that

$$A(D)(l^s_{p,n}(x,\xi)b_{p,n}(y,\eta)) = s(V'_{p,n}(x) + V'_1(x))l^{s-1}_{p,n}(2\xi) - (2s\xi l^{s-1}_{p,n})(V'_{p,n}(x) + V'_1(x)) = 0.$$

This implies that

$$[\mathrm{Op}^w(l^s_{p,n}), \tilde{\Delta}_{p,n}] = 2\left(\sum_{k=1}^{\infty} \frac{(iA(D))^{2k+1}}{(2k+1)!}(l^s_{p,n}(x,\xi)b_{p,n}(y,\eta))\Big|_{x=y,\xi=\eta}\right)^w.$$

Note that $\partial_x\partial_\xi b = 0$ and $\partial^3_\xi b = 0$. Hence,

$$A(D)^{2k+1}l^s_{p,n}(x,\xi)b_{p,n}(y,\eta)\big|_{x=y,\xi=\eta} = \frac{1}{2^{2k+1}}D^{2k+1}_\xi\left(l^s_{p,n}(x,\xi)\right)D^{2k+1}_y\left(b_{p,n}(y,\eta)\right)\Big|_{x=y,\xi=\eta}.$$

Now,

$$|D^{2k+1}_y b_{p,n}(y,\eta)| \leq C_k l_{p,n} \tag{4.3.3}$$

for some constant $C_k$ dependent on $k$. Additionally

$$|D^{2k+1}_\xi l^s_{p,n}(x,\xi)| \leq C_{k,s} l^{s-\frac{2k+1}{2}}_{p,n}. \tag{4.3.4}$$

This is due to fact that we lose one power of $\xi$ with each derivative and

$$|\xi| \le (\xi^2 + C + p^2V_+(x) + n^2V_-(x) + V_1(x))^{1/2}.$$

Combining the estimates in (4.3.3) and (4.3.4) gives

$$\left| A(D)^{2k+1} l_{p,n}^s(x,\xi) b_{p,n}(y,\eta)|_{x=y,\xi=\eta} \right| \le C_{k,s} l_{p,n}^{s-\frac{2k+1}{2}+1}.$$

Since the expansion begins with $k = 1$, we have

$$\left|\left|[\mathrm{Op}^w(l_{p,n}^s), \tilde{\Delta}_{p,n}]u\right|\right|_{L^2} \le C_s ||\mathrm{Op}^w(l_{p,n}^{s-1/2})u|| \le C_s ||\mathrm{Op}^w(l_{p,n}^s)u||_{L^2}.$$

Let $E_{p,n}^s(t) = \int_{\mathbb{R}} |\mathrm{Op}^w(l_{p,n}^{s/2})u_{p,n}|^2 dx$. Then,

$$\begin{aligned}
\frac{d}{dt}E_{p,n}^s(t) =& \int_{\mathbb{R}} (\mathrm{Op}^w(l_{p,n}^{s/2})^{s/2}\partial_t u_{p,n})\mathrm{Op}^w(l_{p,n}^{s/2})\bar{u}_{p,n}dx + \int_{\mathbb{R}} (\mathrm{Op}^w(l_{p,n}^{s/2})u_{p,n})\mathrm{Op}^w(l_{p,n}^{s/2})\partial_t\bar{u}_{p,n}dx \\
=& \int_M \mathrm{Op}^w(l_{p,n}^{s/2})(\tilde{\Delta}_{p,n}u_{p,n})\mathrm{Op}^w(l_{p,n}^{s/2})\bar{u}_{p,n}dx - \int_{\mathbb{R}} \mathrm{Op}^w(l_{p,n}^{s/2})u_{p,n}\mathrm{Op}^w(l_{p,n}^{s/2})(\tilde{\Delta}_{p,n}\bar{u}_{p,n})dx \\
=& 2\,\mathrm{Re}\, i \int_{\mathbb{R}} (\mathrm{Op}^w(l_{p,n}^{s/2})(\tilde{\Delta}_{p,n}u_{p,n}))\mathrm{Op}^w(l_{p,n}^{s/2})\bar{u}_{p,n}dx \\
=& 2\,\mathrm{Re}\, i \int_{\mathbb{R}} [\mathrm{Op}^w(l_{p,n}^{s/2}), \tilde{\Delta}_{p,n}]u_{p,n}\mathrm{Op}^w(l_{p,n}^{s/2})\bar{u}_{p,n}dx \\
\le& \left|\left|[\mathrm{Op}^w(l_{p,n}^{s/2}), \tilde{\Delta}_{p,n}]u_{p,n}\right|\right|_{L^2} \left|\left|\mathrm{Op}^w(l_{p,n}^{s/2})u_{p,n}\right|\right|_{L^2} \\
\le& C_s E_{p,n}^s.
\end{aligned}$$

Following the calculations done in (2.5.3), we get

$$E_{p,n}^s(t) \le C_{t,s} E_{p,n}^s(0)$$

for a constant $C_{t,s}$ dependent on $s$ and $t$. Note that $||\mathrm{Op}^w(l_{p,n}^{s/2})u||$ is equivalent to the $H^s$ norm, so $||u_{p,n}||_{H^s} \le C_{t,s}||u_{p,n,0}||_{H^s}$ for all $t > 0$.

### 4.3.3 Positive commutator argument

We will now do a positive commutator argument at this point. For notational purposes we will drop the $u_{p,n}$ notation and just use $u$. Let $B = f(x,p,n)\partial_x$ where $f$ is a function dependent on $x, p, n$. Then,

$$[\tilde{\Delta}_{p,n}, B] = 2f'(x)\partial_x^2 + f''(x)\partial_x + p^2 f(x)V_+'(x) + n^2 f(x)V_-'(x) + f(x)V_1'(x). \tag{4.3.5}$$

Note that $D_t$ commutes with $B$ and $(D_t - \tilde{\Delta}_{p,n})u = 0$, so

$$[-\tilde{\Delta}_{p,n}, B]u = [D_t - \tilde{\Delta}_{p,n}, B]u = (D_t - \tilde{\Delta}_{p,n})Bu - B(D_t - \tilde{\Delta}_{p,n})u = (D_t - \tilde{\Delta}_{p,n})Bu. \tag{4.3.6}$$

Additionally,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} (D_t - \tilde{\Delta}_{p,n})Bu\bar{u}dxdt \\
&\qquad = \int_0^T \int_{\mathbb{R}} -\tilde{\Delta}_{p,n}Bu\bar{u}dxdt + \frac{1}{i}\left[\int_{\mathbb{R}} Bu\bar{u}dx\right]_0^T + \int_0^T \int_{\mathbb{R}} Bu\overline{D_t u}dxdt \\
&\qquad = \frac{1}{i}\left[\int_{\mathbb{R}} Bu\bar{u}dx\right]_0^T - \int_0^T \int_{\mathbb{R}} Bu\overline{(\tilde{\Delta}_{p,n}u)}dxdt + \int_0^T \int_{\mathbb{R}} Bu\overline{D_t u}dxdt \\
&\qquad = \frac{1}{i}\left[\int_{\mathbb{R}} Bu\bar{u}dx\right]_0^T + \int_0^T \int_{\mathbb{R}} Bu\overline{(D_t - \tilde{\Delta}_{p,n})u}dxdt \\
&\qquad = \frac{1}{i}\left[\int_{\mathbb{R}} Bu\bar{u}dx\right]_0^T.
\end{aligned}$$

Using (4.3.6) gives

$$\int_0^T \int_{\mathbb{R}} [-\tilde{\Delta}_{p,n}, B]u\bar{u}dxdt = \frac{1}{i}\left[\int_{\mathbb{R}} Bu\bar{u}dx\right]_0^T.$$

From (4.3.5)

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} -\Big(\big(2f'(x)\partial_x^2 + f''(x)\partial_x + p^2 f(x)V_+'(x) + n^2 f(x)V_-'(x)\big)u\Big)\bar{u}dxdt \\
&\qquad = \frac{1}{i}\left[\int_{\mathbb{R}} Bu\bar{u}dx\right]_0^T + \int_0^T \int_{\mathbb{R}} f(x)V_1'(x)u\bar{u}dxdt.
\end{aligned}$$

Integrating by parts and moving over the term with $f''(x)$ gives

$$\int_0^T \int_{\mathbb{R}} 2f'(x)(\partial_x u)\overline{\partial_x u} - p^2 f(x)V_+'(x)u\bar{u} - n^2 f(x)V_-'(x)u\bar{u}dxdt$$
$$= \frac{1}{i}\left[\int_{\mathbb{R}} Bu\bar{u}dx\right]_0^T + \int_0^T \int_{\mathbb{R}} f(x)V_1'(x)u\bar{u}dxdt - \int_0^T \int_{\mathbb{R}} f''(x)\partial_x u\bar{u}dxdt.$$

We want to estimate the absolute value of the right hand side where $f(x) \in \mathcal{C}^\infty$ such that $f$ and all of its derivatives are bounded. Recall from the proof of local smoothing in $\mathbb{R}^2$ and (3.2.7) that $\left|\langle f(z)\partial_z u, u\rangle\right| \leq C||u_0||^2_{H^{1/2}}$ for $f(z) \in \mathcal{C}^\infty$ where $f$ and its derivatives are bounded. This immediately gives that

$$\left|\frac{1}{i}\left[\int_{\mathbb{R}} Bu\bar{u}dx\right]_0^T\right| \leq C_T||u_0||^2_{H^{1/2}} \tag{4.3.7}$$

and

$$\left|\int_0^T \int_{\mathbb{R}} f''(x)\partial_x u\bar{u}dxdt\right| \leq C_T||u_0||^2_{H^{1/2}} \tag{4.3.8}$$

for some constant $C_T$. To handle the last term recall that

$$V_1(x) = \frac{U_+^2(U_-')^2 + (U_+')^2U_-^2 - 2U_+^2U_-''U_- - 2U_+''U_+U_-^2 - 2U_+U_-U_+'U_-'}{4U_+^2U_-^2}.$$

Note that $U_+, U_- \geq \frac{1}{\sqrt{\tilde{M}}}$, $U_+, U_- \in \mathcal{C}^\infty$. Additionally $U_\pm(x) = |x|$ outside of a compact set so that $V_1(x) \to 0$ as $x \to \pm\infty$. Hence, $V_1'(x) \in \mathcal{C}^\infty$ and $V_1'(x)$ and its derivatives are bounded. We get

$$\left|\int_0^T \int_{\mathbb{R}} f(x)V_1'(x)u\bar{u}dxdt\right| \leq C_T'||u_0||^2_{H^{1/2}} \tag{4.3.9}$$

for some constant $C_T'$. Combining the estimates in (4.3.7), (4.3.8), and (4.3.9) gives

$$\int_0^T \langle f'(x)\partial_x u, \partial_x u\rangle + \langle -(p^2 f(x)V_+'(x) + n^2 f(x)V_-'(x))u, u\rangle dt \leq (2C_T + C_T')||u_0||^2_{H^{1/2}},$$

if $-(p^2 f(x,p,n)V_+'(x) + n^2 f(x,p,n)V_-'(x)) \geq 0$ and $f'(x,p,n) \geq 0$. To figure out the right function $f(x,p,n)$ we have to understand the critical points for the potential $V_{p,n}(x) = p^2V_+(x) + n^2V_-(x)$. We will first show that the only critical point is in the interval $[-\varepsilon, \varepsilon]$. We have

$$V_+' = K'(x)(\tilde{M} - (x-\varepsilon)^{2m_1}) - 2m_1(x-\varepsilon)^{2m_1-1}K(x) - K'(x)x^{-2} + (1-K(x))(-2x^{-3})$$

and

$$V_-' = K'(x)(\tilde{M} - (x+\varepsilon)^{2m_2}) - 2m_2(x+\varepsilon)^{2m_2-1}K(x) - K'(x)x^{-2} + (1-K(x))(-2x^{-3}).$$

If $|x| \leq 2\varepsilon$, we have $K'(x) = 0$ and $(1-K(x)) = 0$. Hence,

$$V_+' = -2m_1(x-\varepsilon)^{2m_1-1}, V_-' = -2m_2(x+\varepsilon)^{2m_2-1}.$$

If $\varepsilon < x \leq 2\varepsilon$, then

$$V_\pm'(x) < 0.$$

If $-2\varepsilon \leq x < \varepsilon$, then

$$V_\pm'(x) > 0.$$

If $|x| \geq 4\varepsilon$, we have $K'(x) = 0$ and $K(x) = 0$. Hence,

$$V_\pm' = -2x^{-3}. \tag{4.3.10}$$

If $-4\varepsilon < x < -2\varepsilon$, then $K'(x) \geq 0$. So, $K'(x)(\tilde{M} - (x-\varepsilon)^{2m_1}) \geq 0$ and $K'(x)(\tilde{M} - (x+\varepsilon)^{2m_2}) \geq 0$, since $\tilde{M}$ was chosen so that $(\tilde{M} - (x-\varepsilon)^{2m_1}) > 0$ and $(\tilde{M} - (x+\varepsilon)^{2m_2}) > 0$ on $\operatorname{supp}(K)$. Additionally, $(-2m_1(x-\varepsilon)^{2m_1-1})K(x) \geq 0, (-2m_2(x+\varepsilon)^{2m_2-1})K(x) \geq 0, K'(x)(x^{-2}) \geq 0$ and $(1-K(x))(-2x^{-3}) \geq 0$. Furthermore, we must have $(-2m_1(x-\varepsilon)^{2m_1-1})K(x) > 0$ or $(1-K(x))(-2x^{-3}) > 0$ and $(-2m_2(x+\varepsilon)^{2m_2-1})K(x) > 0$ or $(1-$

$K(x))(-2x^{-3}) > 0$ on this range. Hence,

$$V'_{\pm} \geq C \tag{4.3.11}$$

for some constant $C > 0$.

If $4\varepsilon > x > 2\varepsilon$, then $K'(x) \leq 0$. So, $K'(x)(\tilde{M}-(x-\varepsilon)^{2m_1}) \leq 0$ and $K'(x)(\tilde{M}-(x+\varepsilon)^{2m_2}) \leq 0$, since $\tilde{M}$ was chosen so that $(\tilde{M} - (x-\varepsilon)^{2m_1}) > 0$ and $(\tilde{M} - (x+\varepsilon)^{2m_2}) > 0$ on $\operatorname{supp}(K)$. Additionally, $(-2m(x \pm \varepsilon)^{2m-1})K(x) \leq 0, K'(x)(x^{-2}) \leq 0$ and $(1 - K(x))(-2x^{-3}) \leq 0$. Furthermore, we must have $(-2m_1(x-\varepsilon)^{2m_1-1})K(x) < 0$ or $(1-K(x))(-2x^{-3}) < 0$ and $(-2m_2(x+\varepsilon)^{2m_2-1})K(x) < 0$ or $(1-K(x))(-2x^{-3}) < 0$ on this range. Hence,

$$V'_{\pm} \leq -C \tag{4.3.12}$$

for some constant $C > 0$.

Combining the estimates above for when $x > \varepsilon$ or $x < -\varepsilon$ gives that $V'_+(x)$ and $V'_-(x)$ have the same sign for $x \notin [-\varepsilon, \varepsilon]$ and that $V'(x) \neq 0$. Since on $[-\varepsilon, \varepsilon]$ $V''_{\pm} \leq 0$ for each $p$ and $n$, there is only a single critical, which will be denoted $x_{p,n}$, for $V_{p,n}(x) = p^2 V_+(x) + n^2 V_-(x)$ and $x_{p,n} \in [-\varepsilon, \varepsilon]$.

Next, define

$$\alpha(x) = \int_0^x \Upsilon(y)dy$$

for

$$\Upsilon(x) = \begin{cases} 1, & |x| \leq 3\varepsilon \\ C_\Upsilon/|x|^3 & |x| > 4\varepsilon \end{cases}$$

such that that $\Upsilon(x)$ is smooth and even. This implies that $\alpha(x) = x$ when $|x| < 3\varepsilon$, $|\alpha(x)|$ is bounded, and $\alpha$ is an odd function. This tells us that by taking $f(x,p,n) = \alpha(x - x_{p,n})$,

$-(p^2 f(x)V_+'(x) + n^2 f(x)V_-'(x)) \geq 0$ for all $x$. Hence,

$$\int_0^T ||\Upsilon^{1/2}(x)\partial_x u||_{L^2}^2 + \langle -(p^2 f(x)V_+'(x) + n^2 f(x)V_-'(x))u, u\rangle dt \leq C_T ||u_0||_{H^{1/2}}^2. \quad (4.3.13)$$

Next, we want to get a lower bound on

$$\int_0^T \langle -(p^2 f(x)V_+'(x) + n^2 f(x)V_-'(x))u, u\rangle dt.$$

Using the estimates (4.3.10), (4.3.11) and (4.3.12) on $V_\pm'$ we have,

$$\begin{aligned}
&\int_0^T - \langle (p^2 f(x)V_+'(x) + n^2 f(x)V_-'(x))u, u\rangle dt \\
&\geq C_1 \int_0^T \int_{\mathbb{R}\setminus[-4\varepsilon,4\varepsilon]} (p^2+n^2)|x|^{-3}|u|^2 dxdt + C_2 \int_0^T \int_{[-4\varepsilon,-2\varepsilon]\cup[2\varepsilon,4\varepsilon]} (p^2+n^2)|u|^2 dxdt \\
&\quad + \int_0^T \int_{[-2\varepsilon,2\varepsilon]} -(p^2 f(x)V_+'(x) + n^2 f(x)V_-'(x))|u|^2 dxdt
\end{aligned} \quad (4.3.14)$$

for some constants $C_1, C_2 > 0$ and independent of $n$ and $p$. We will estimate

$$\int_0^T \int_{[-2\varepsilon,2\varepsilon]} -(p^2 f(x)V_+'(x) + n^2 f(x)V_-'(x))|u|^2 dxdt$$

in the following section.

### 4.3.4 Estimates on the Potentials

The goal of this section is to prove a lower bound estimate of $-(p^2 f(x)V_+'(x)+n^2 f(x)V_-'(x))$.

**Proposition 4.3.15.** *Let $P(x) = p^2(x-\varepsilon)^{2m_1} + n^2(x+\varepsilon)^{2m_2}$ for positive integers $p, n, m_1, m_2$. Let $x_0$ be the critical point of $P(x)$. Then,*

$$P'(x)(x-x_0) \geq C_{\varepsilon,m_1,m_2}\left[(p^2(x-x_0)^{2m_1} + n^2(x-x_0)^{2m_2} + \min\{p^2,n^2\}(x-x_0)^2\right]$$

*for a constant $C_{\varepsilon,m_1,m_2} > 0$ dependent on $\varepsilon$, $m_1$, and $m_2$, but independent of $p, n$.*

With Proposition 4.3.15 we have that

$$\int_0^T \int_{[-2\varepsilon,2\varepsilon]} -(p^2 f(x)V_+'(x) + n^2 f(x)V_-'(x))|u|^2 dxdt$$
$$\geq \int_0^T \int_{[-2\varepsilon,2\varepsilon]} C_{\varepsilon,m_1,m_2}\left[(p^2(x-x_0)^{2m_1} + n^2(x-x_0)^{2m_2} + \min\{p^2,n^2\}(x-x_0)^2\right]|u|^2 dxdt, \tag{4.3.16}$$

since $-(p^2 f(x)V_+'(x) + n^2 f(x)V_-'(x)) = P'(x)(x-x_0)$ when $|x| \leq 2\varepsilon$.

**Remark 4.3.17.** *This estimate tells us that if $p \gg n$ or $n \gg p$, then our estimate is similar to the $x^{2m_1}$ case or $x^{2m_2}$ case in [CW13] respectively. When $p \sim n$, the estimate is similar to the $x^2$ case in [Chr08].*

We will prove Proposition 4.3.15 by using Lemmas 4.3.18 and 4.3.19.

**Lemma 4.3.18.** *Let $P(x) = p^2(x-\varepsilon)^{2m_1} + n^2(x+\varepsilon)^{2m_2}$ where $p, n, m_1, m_2$ are positive integers and $x_0$ is the critical point of $P(x)$. Then,*

$$P'(x)(x-x_0) \geq C_{\varepsilon,m_1,m_2} \min\{p^2,n^2\}(x-x_0)^2$$

*for a constant $C_{\varepsilon,m_1,m_2} > 0$ dependent on $\varepsilon$, $m_1$ and $m_2$, but independent of $p$ and $n$.*

**Lemma 4.3.19.** *Let $P(x) = p^2(x-\varepsilon)^{2m_1} + n^2(x+\varepsilon)^{2m_2}$ for positive integers $p, n, m_1, m_2$. Let $x_0$ be the critical point of $P(x)$. Let $|x| \leq 2\varepsilon$. Then,*

$$P'(x)(x-x_0) \geq C(p^2(x-x_0)^{2m_1} + n^2(x-x_0)^{2m_2})$$

*for a positive constant $C$ independent of $p$ and $n$.*

Lemma 4.3.19 will follow from breaking the estimate up into the cases when $|x| \leq \varepsilon$ and when $|x| > \varepsilon$. We will need Lemma 4.3.20 for the case when $|x| \leq \varepsilon$ to prove Lemma 4.3.19.

**Lemma 4.3.20.** *Let $Y, Z > 0$ with $Z > Y$ and $m$ a positive integer. Then,*

$$(Y - Z)^{2m-1} \geq Y^{2m-1} - Z^{2m-1}.$$

**Remark 4.3.21.** *Note that $Y^{2m-1} - Z^{2m-1}$ is negative, so we will have to be careful using this estimate with inequalities.*

We will need Lemma 4.3.22 for when $|x| > \varepsilon$ to prove Lemma 4.3.19.

**Lemma 4.3.22.** *Let $1 > Y > -1$, $2 \geq Z > 1$ and $m_1$ and $m_2$ be positive integer. There exists a $C > 0$ such that*

$$(Y + 1)^{2m_2-1}(Z - 1)^{2m_1-1} + (Z + 1)^{2m_2-1}(1 - Y)^{2m_1-1} \geq C(Z - Y)^{2m_1-1}.$$

Once we have both lemmas, we can combine the results from Lemma 4.3.18 and 4.3.19 to get Proposition 4.3.15.

We will begin by proving Lemmas 4.3.20 and 4.3.22, since they will be needed for Lemma 4.3.19.

*Proof of Lemma 4.3.20.* We will do a proof by induction. Recall that for this lemma $Y, Z > 0$ with $Z > Y$ and $m$ a positive integer. For $m = 2$,

$$\begin{aligned}(Y - Z)^3 &= Y^3 - 3Y^2Z + 3YZ^2 - Z^3 \\ &\geq Y^3 - Z^3,\end{aligned}$$

since $Y^2Z < YZ^2$.

Suppose $(Y-Z)^{2m-1} \geq Y^{2m-1} - Z^{2m-1}$ holds for arbitrary $m \geq 2$. Then,

$$\begin{aligned}
(Y-Z)^{2m+1} &= (Y-Z)^{2m-1}(Y-Z)^2 \\
&\geq (Y^{2m-1} - Z^{2m-1})(Y-Z)^2 \\
&= (Y^{2m-1} - Z^{2m-1})(Y^2 - 2YZ + Z^2) \\
&= Y^{2m+1} - 2Y^{2m}Z + Y^{2m-1}Z^2 - Z^{2m-1}Y^2 + 2YZ^{2m} - Z^{2m+1} \\
&= Y^{2m+1} - Z^{2m+1} + (-2Y^{2m}Z + Y^{2m-1}Z^2 - Z^{2m-1}Y^2 + 2YZ^{2m}) \\
&\geq Y^{2m+1} - Z^{2m+1}
\end{aligned}$$

if $-2Y^{2m}Z + Y^{2m-1}Z^2 - Z^{2m-1}Y^2 + 2YZ^{2m} \geq 0$. For $Y, Z > 0$ and $Z > Y$

$$\begin{aligned}
-2Y^{2m}Z + Y^{2m-1}Z^2 - Z^{2m-1}Y^2 + 2YZ^{2m} &= YZ(2Z^{2m-1} + Y^{2m-2}Z - 2Y^{2m-1} - Z^{2m-2}Y) \\
&= YZ\big(Z(2Z^{2m-2} + Y^{2m-2}) - Y(2Y^{2m-2} + Z^{2m-2})\big) \geq 0.
\end{aligned}$$

□

*Proof of Lemma 4.3.22.* Recall we have that $1 > Y > -1$, $2 \geq Z > 1$, and $m_1$ and $m_2$ are positive integers. Let $Z - Y = \delta$ and $1 - Y = c\delta$, so that $Z - 1 = (1-c)\delta$. Note that $(1-c) > 0$, $c > 0$, and $\delta > 0$. Then,

$$\begin{aligned}
(Z-Y)^{-(2m_1-1)}&\Big[(Y+1)^{2m_2-1}(Z-1)^{2m_1-1} + (Z+1)^{2m_2-1}(1-Y)^{2m_1-1}\Big] \\
&= (Y+1)^{2m_2-1}(1-c)^{2m_1-1} + (Z+1)^{2m_2-1}c^{2m_1-1}.
\end{aligned}$$

If $Y \geq 0$, then

$$(Y+1)^{2m_2-1}(1-c)^{2m_1-1} + (Z+1)^{2m_2-1}c^{2m_1-1} \geq (1-c)^{2m_1-1} + c^{2m_2-1} \geq C \tag{4.3.23}$$

for a constant $C > 0$ independent of $Y, Z$. If $Y \leq 0$, then $c \geq \frac{1}{2}$, since $Z \leq 2$. Using this and that $Z > 1$ gives

$$(Y+1)^{2m_2-1}(1-c)^{2m_1-1} + (Z+1)^{2m_2-1}c^{2m_1-1} \geq 2^{2m_2-1}2^{-(2m_1-1)} \geq 2^{2m_2-2m_1}. \quad (4.3.24)$$

Combining the estimates from (4.3.23) and (4.3.24) gives that there is a constant $C > 0$ such that

$$(Z-Y)^{-(2m_1-1)}[(Y+1)^{2m_2-1}(Z-1)^{2m_1} + (Z+1)^{2m_2-1}(1-Y)^{2m_1-1}] \geq C \quad (4.3.25)$$

for $-1 < Y < 1$. Multiplying by $(Z-Y)^{(2m_1-1)}$ on both sides of (4.3.25) gives

$$(Y+1)^{2m_2-1}(Z-1)^{2m_1-1} + (Z+1)^{2m_2-1}(1-Y)^{2m_1-1} \geq C(Z-Y)^{2m_1-1}$$

as desired. □

*Proof of Lemma 4.3.19.* Recall

$$P(x) = p^2(x-\varepsilon)^{2m_1} + n^2(x+\varepsilon)^{2m_2}.$$

From the definition of $P(x)$ note that the critical point $x_0$ will satisfy,

$$2m_1p^2(x_0-\varepsilon)^{2m_1-1} + 2m_2n^2(x_0+\varepsilon)^{2m_2-1} = 0. \quad (4.3.26)$$

Calculating $P'(x)(x-x_0)$ gives

$$\begin{aligned} P'(x)(x-x_0) &= \left[2m_1p^2(x-\varepsilon)^{2m_1-1} + 2m_2n^2(x+\varepsilon)^{2m_2-1}\right](x-x_0) \\ &= \left[2m_1p^2((x-x_0)+(x_0-\varepsilon))^{2m_1-1} + 2m_2n^2((x-x_0)+(x_0+\varepsilon))^{2m_2-1}\right](x-x_0). \end{aligned}$$

Note that $(x_0-\varepsilon) \leq 0$ and $(x_0+\varepsilon) \geq 0$. This implies if $x < x_0$, then $((x-x_0)+(x_0-\varepsilon))^{2m_1-1}(x-x_0) \geq (x-x_0)^{2m_1} + (x_0-\varepsilon)^{2m_1-1}(x-x_0)$ holds, since both $(x-x_0) < 0$ and $(x_0-\varepsilon) < 0$.

Similarly, if $x > x_0$, then $((x-x_0)+(x_0+\varepsilon))^{2m_2-1}(x-x_0) \geq (x-x_0)^{2m_2}+(x_0+\varepsilon)^{2m_2-1}(x-x_0)$ holds, since both $(x - x_0) > 0$ and $(x_0 + \varepsilon) > 0$.

Note that if $x = \varepsilon$, then

$$((x - x_0) + (x_0 - \varepsilon))^{2m_1-1}(x - x_0) = 0 = (x - x_0)^{2m_1} + (x_0 - \varepsilon)^{2m_1-1}(x - x_0). \quad (4.3.27)$$

Additionally, if $x = -\varepsilon$, then

$$((x - x_0) + (x_0 + \varepsilon))^{2m_w-1}(x - x_0) = 0 = (x - x_0)^{2m_1} + (x_0 - \varepsilon)^{2m_2-1}(x - x_0). \quad (4.3.28)$$

If $\varepsilon \geq x > x_0$, then using the Lemma 4.3.20 and (4.3.27)

$$((x - x_0) + (x_0 - \varepsilon))^{2m_1-1}(x - x_0) \geq (x - x_0)^{2m_1} + (x_0 - \varepsilon)^{2m_1-1}(x - x_0).$$

Note that $x - x_0$ is positive, so even though $(x - x_0)^{2m_1-1} + (x_0 - \varepsilon)^{2m_1-1}$ is negative the inequality holds. Similarly, if $-\varepsilon \leq x < x_0$, then using Lemma 4.3.20 with $Y = (x - x_0)$, $Z = -(\varepsilon - x_0)$ and (4.3.28) gives

$$\begin{aligned}((x - x_0) + (x_0 + \varepsilon))^{2m_2-1}(x - x_0) &= ((x_0 - x) + (-\varepsilon - x_0))^{2m_2-1}(x_0 - x)\\ &\geq (x_0 - x)^{2m_2} + (-\varepsilon - x_0)^{2m_2-1}(x_0 - x)\\ &= (x - x_0)^{2m_2} + (x_0 + \varepsilon)^{2m_2-1}(x - x_0).\end{aligned}$$

Combining these results gives

$$
\begin{aligned}
&P'(x)(x-x_0)\\
=&\big[2m_1p^2((x-x_0)+(x_0-\varepsilon))^{2m_1-1}+2m_2n^2((x-x_0)+(x_0+\varepsilon))^{2m_2-1}\big](x-x_0)\\
\geq&2m_1p^2[(x-x_0)^{2m_1}+(x_0-\varepsilon)^{2m_1-1}(x-x_0)]+2m_2n^2[(x-x_0)^{2m_2}+(x_0+\varepsilon)^{2m_2-1}(x-x_0)]\\
=&2(m_1p^2(x-x_0)^{2m_1}+m_2n^2(x-x_0)^{2m_2}+[m_1p^2(x_0-\varepsilon)^{2m_1-1}+m_2n^2(x_0+\varepsilon)^{2m_2-1}](x-x_0))\\
=&2m_1p^2(x-x_0)^{2m_1}+2m_2n^2(x-x_0)^{2m_2},\ \text{by equation (4.3.26)} \qquad (4.3.29)
\end{aligned}
$$

as desired for $-\varepsilon \leq x \leq \varepsilon$.

Now, we need to handle the case when $|x| \geq \varepsilon$. First we will consider $x > \varepsilon$. In this case

$$P'(x)(x-x_0) \geq 2m_2n^2(x-x_0)^{2m_2} \qquad (4.3.30)$$

since $x+\varepsilon > x-x_0$. Our goal will be to show

$$P'(x)(x-x_0) \geq Cp^2(x-x_0)^{2m_1-1}(x-x_0) \qquad (4.3.31)$$

for $2\varepsilon \geq x > \varepsilon$. The idea will be to replace $n$ in the equation for $P'(x)$ with $p$. We will use (4.3.26) to get

$$m_2n^2 = -2m_1p^2(x_0-\varepsilon)^{2m_1-1}(x_0+\varepsilon)^{-(2m_2-1)}. \qquad (4.3.32)$$

Substituting (4.3.32) into the equation for $P'(x)$ gives

$$
\begin{aligned}
P'(x) &= 2m_1p^2((x-\varepsilon)^{2m_1-1}-(x+\varepsilon)^{2m_2-1}(x_0-\varepsilon)^{2m_1-1}(x_0+\varepsilon)^{-(2m_2-1)})\\
&= \frac{2m_1p^2}{(x_0+\varepsilon)^{2m_2-1}}((x-\varepsilon)^{2m_1-1}(x_0+\varepsilon)^{2m_2-1}+(x+\varepsilon)^{2m_2-1}(\varepsilon-x_0)^{2m_1-1}). \qquad (4.3.33)
\end{aligned}
$$

The case for $x_0=-\varepsilon$ is when $p=0$ and the case for $x_0=\varepsilon$ is when $n=0$ and $x-\varepsilon=x-x_0$. In both cases (4.3.31) holds. Now, let $x_0=Y\varepsilon$ and $x=Z\varepsilon$ for $1>Y>-1$ and $2\geq Z>1$.

Then,

$$P'(x) = \frac{2m_1p^2\varepsilon^{2m_1-1}}{(Y+1)^{2m_2-1}}((Y+1)^{2m_2-1}(Z-1)^{2m_1-1} + (Z+1)^{2m_2-1}(1-Y)^{2m_1-1}) \quad (4.3.34)$$

Note that $x - x_0 = (Z-Y)\varepsilon$. Using Lemma 4.3.22 and noting $0 < (Y+1) \leq 2$ gives

$$\begin{aligned} P'(x)(x-x_0) &\geq C\frac{2m_1p^2\varepsilon^{2m_1-1}}{2^{2m_2-1}}(Z-Y)^{2m-1}(x-x_0) \\ &\geq C\frac{2m_1}{2^{2m_2-1}}p^2(x-x_0)^{2m_1} \end{aligned} \quad (4.3.35)$$

for a constant $C > 0$. Hence,

$$P'(x)(x-x_0) \geq Cp^2(x-x_0)^{2m_1} \quad (4.3.36)$$

as desired for $C > 0$ independent of $p$ and $n$. Combining (4.3.36) and (4.3.30) gives

$$P'(x)(x-x_0) \geq C(p^2(x-x_0)^{2m_1} + 2m_2n^2(x-x_0)^{2m_2}) \quad (4.3.37)$$

for $2\varepsilon \geq x > \varepsilon$.

If $x < -\varepsilon$, then $(x-\varepsilon) < (x-x_0) < 0$. So,

$$P'(x)(x-x_0) = 2m_1p^2(x-\varepsilon)^{2m_1-1}(x-x_0) \geq 2m_1p^2(x-x_0)^{2m_1}. \quad (4.3.38)$$

Our goal will be to show

$$P'(x) = 2m_1p^2(x-\varepsilon)^{2m_1-1} + 2m_2n^2(x+\varepsilon)^{2m_2-1} \leq Cn^2(x-x_0)^{2m_2-1} \quad (4.3.39)$$

since $P'(x) < 0$ and $(x-x_0) < 0$. Let $y = -x$ and $y_0 = -x_0$. Then, (4.3.39) is equivalent to showing

$$P'(y) = 2m_1p^2(y+\varepsilon)^{2m_1-1} + 2m_2n^2(y-\varepsilon)^{2m_2-1} \geq Cn^2(y-y_0)^{2m_2-1} \tag{4.3.40}$$

for $2\varepsilon \geq y > \varepsilon$. Now (4.3.36) tells us that (4.3.40) holds by swapping $m_2$ with $m_1$ and $p$ with $n$. Combining (4.3.40) and (4.3.38) gives

$$P'(x)(x-x_0) \geq C(p^2(x-x_0)^{2m_1} + n^2(x-x_0)^{2m_2}) \tag{4.3.41}$$

for $-2\varepsilon \leq x < -\varepsilon$ for a constant $C > 0$ independent of $p$ and $n$. Combining the estimates from (4.3.29), (4.3.37) and (4.3.41) gives

$$P'(x)(x-x) \geq C(p^2(x-x_0)^{2m_1} + n^2(x-x_0)^{2m_2})$$

for $|x| \leq 2\varepsilon$ for a constant $C > 0$ independent of $p$ and $n$. □

*Proof of Lemma 4.3.18.* Recall $P(x) = p^2(x-\varepsilon)^{2m_1} + n^2(x+\varepsilon)^{2m_2}$. Let $m = \min(m_1, m_2)$. Using the Taylor approximation theorem gives

$$\begin{aligned}
P'(x)(x-x_0) &= [P'(x_0) + P''(\tilde{x})(x-x_0)](x-x_0), \text{ for } \tilde{x} \text{ between } x_0 \text{ and } x \\
&= [2m_1(2m_1-1)p^2(\tilde{x}-\varepsilon)^{2m_1-2} + 2m_2(2m_2-1)n^2(\tilde{x}+\varepsilon)^{2m_2-2}](x-x_0)^2 \\
&\geq 2m(2m-1)\min\{n^2,p^2\}[(\tilde{x}-\varepsilon)^{2m_1-2} + (\tilde{x}+\varepsilon)^{2m_2-2}](x-x_0)^2 \\
&\geq 2m(2m-1)\min\{n^2,p^2\}\min\{\varepsilon^{2m_1-2}, \varepsilon^{2m_2-2}\}(x-x_0)^2 \\
&= C_{\varepsilon,m_1,m_2}\min\{n^2,p^2\}(x-x_0)^2
\end{aligned}$$

for a positive constant $C_{\varepsilon,m_1,m_2}$ independent of $p$ and $n$. □

### 4.3.5 Summary of local smoothing estimates

Piecing together the estimates we get perfect local smoothing in the $x$-direction. Furthermore we get perfect local smoothing away from a neighborhood of $[-\varepsilon, \varepsilon]$ in the $\theta_+, \theta_-$ directions. Combining the estimates in (4.3.13), (4.3.14), and (4.3.16) gives Lemma 4.3.42.

**Lemma 4.3.42** (Local Smoothing away from trapping)**.** *For any* $\delta > 0$

$$\int_0^T ||\langle x\rangle^{-3/2}\partial_x u||^2_{L^2} + ||\langle x\rangle^{-3/2}\chi_\delta(x)\partial_{\theta_+}u||^2_{L^2} + ||\langle x\rangle^{-3/2}\chi_\delta(x)\partial_{\theta_-}u||^2_{L^2} dt \leq C_{T,\delta,\varepsilon}||u_0||^2_{H^{1/2}}$$

*where* $\chi_\delta(x)$ *is a cutoff function satisfying* $\chi_\delta(x) = 0$ *for* $x \in [-\varepsilon - \delta, \varepsilon + \delta]$ *and* $\chi_\delta(x) = 1$ *for* $x \notin [-\varepsilon - 2\delta, \varepsilon + 2\delta]$*. The* $L^2$ *and* $H^{1/2}$ *norms are in the variables* $x, \theta_-$ *and* $\theta_+$*.*

What is shown is slightly stronger. It says that for each frequency $u_{p,n}$ there is local smoothing away from the critical point, $x_{p,n}$, of the potential $p^2V_+(x) + n^2V_-(x)$. Formally,

$$\begin{aligned}
&\int_0^T ||\langle x\rangle^{-3/2}\partial_x u||^2_{L^2(x,\theta_+,\theta_-)} dt + \sum_{p,n}^{\infty}\left[\int_0^T ||\langle x\rangle^{-3/2-1}(x - x_{p,n})\min\{p,n\}u_{p,n}||^2_{L^2(x)} dt\right] \\
&+ \sum_{p,n}^{\infty}\left[\int_0^T ||\langle x\rangle^{-3/2-m_1}(x - x_{p,n})^{m_1} p u_{p,n}]||^2_{L^2(x)} + ||\langle x\rangle^{-3/2-m_2}(x - x_{p,n})^{m_2} n u_{p,n}||^2_{L^2(x)} dt\right] \\
&\leq C_{T,\varepsilon,m}||u_0||^2_{H^{1/2}(x,\theta_+,\theta_-)}.
\end{aligned}$$

Now let $m_1 = m_2$. Ignoring the possible better estimates from the

$$\int_0^T ||\langle x\rangle^{-3/2}\min\{p,n\}(x - x_{p,n})u||^2_{L^2} dt$$

we expect that due to the localizing term of $(x - x_{p,n})^m$ that

$$\begin{aligned}
&\int_0^T ||\chi(x)p u_{p,n}||^2_{L^2} + ||\chi(x) n u_{p,n}||^2_{L^2} dt \\
&\qquad \leq C_T(||\langle p\rangle^{m/(m+1)}u_{0,p,n}||^2_{L^2} + ||\langle n\rangle^{m/(m+1)}u_{0,p,n}||^2_{L^2} + ||u_{0,p,n}||^2_{H^{1/2}}) \quad (4.3.43)
\end{aligned}$$

to match with the estimate in [CW13] where $\chi(x) \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\chi(x) \equiv 1$ on $[-\varepsilon, \varepsilon]$.

Set $h^{-2} = p^2 + n^2$ and $\tilde{u}_{p,n}(x) = u_{p,n}(x + x_{p,n})$. Then, $\tilde{u}_{p,n}(x)$ solves

$$(D_t + D_x^2 + p^2 V_+(x + x_{p,n}) + n^2 V_-(x + x_{p,n}) + V_1(x + x_{p,n}))\tilde{u}_{p,n} = 0.$$

Let $\tilde{V}_{p,n}(x) = p^2 V_+(x + x_{p,n}) + n^2 V_-(x + x_{p,n})$ and $\tilde{V}_1(x) = V_1(x + x_{p,n})$. We have $\tilde{V}'_{p,n}(0) = 0$ and $\tilde{V}'_{p,n}(x) \geq h^{-2} x^{2m}$. Since this is just shifting $u_{p,n}$,

$$\sum_{p,n}^{\infty} \left[ \int_0^T ||\langle x\rangle^{-3/2} \partial_x \tilde{u}_{p,n}||_{L^2}^2 dt + \int_0^T ||\langle x\rangle^{-3/2-m} p x^m \tilde{u}_{p,n}||_{L^2}^2 + ||\langle x\rangle^{-3/2-m} n x^m \tilde{u}_{p,n}||_{L^2}^2 dt \right]$$
$$+ \sum_{p,n}^{\infty} \left[ \int_0^T ||\langle x\rangle^{-3/2-1} \min\{p, n\} x \tilde{u}_{p,n}||_{L^2}^2 dt \right] \leq C_{T,\varepsilon,m} ||u_0||_{H^{1/2}}^2.$$

Using the new notation (4.3.43) holds, if

$$\int_0^T ||\chi(x) h^{-1} \tilde{u}_{p,n}(x)||_{L^2}^2 dt \leq C_T(||\langle h^{-1}\rangle^{m/(m+1)} u_{p,n,0}||_{L^2}^2 + ||u_{p,n,0}||_{H^{1/2}}^2) \tag{4.3.44}$$

where $\chi(x) \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\chi(x) \equiv 1$ near $x = 0$. The key reason for this change is that it allows us to consider $p, n$ at the same time and makes the critical point for $\tilde{V}_{p,n}$ at $x = 0$ for all $p, n$. This means that when $h \to 0$ the critical point is fixed.

## 4.4 Estimating at the critical point

### 4.4.1 Low Frequency Estimate

We want to break up $u$ into high and low angular frequency parts. The idea is that if the angular frequency is low compared to the radial frequency, then we can estimate the angular derivative by the radial derivative, which we have an estimate for, and an error term which is estimated similar to the high frequency part. In this situation we have two angular directions and both affect the nature of the trapping. This issue is one of the reasons to consider $h^{-2} = p^2 + n^2$. If $h$ is small, then the total angular frequency is large. Next, we will formalize the decomposition into high and low frequency parts and show that the high frequency estimate bounds the low frequency estimate.

Define

$$u_{hi} = \psi(hD_x)\tilde{u}, \;\; u_{lo} = (1-\psi(hD_x))\tilde{u}$$

where $\psi(r) \in \mathcal{C}_c^\infty(\mathbb{R})$ which is 1 for $|r| \leq \varepsilon'$ and vanishes for $|r| \leq 2\varepsilon'$ for $\varepsilon' > 0$ small. Then,

$$(D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1)u_{lo} = [D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1, (1-\psi)]\tilde{u} = [\tilde{V}_{p,n} + \tilde{V}_1, -\psi]\tilde{u} = h^{-1}L_1(\tilde{u}). \tag{4.4.1}$$

Since $\tilde{V}_{p,n} = \mathcal{O}(h^{-2})$, when we take the commutator $L_1$ is a pseudo-differential operator (with parameter $h$) of order zero such that the support of the symbol of $L_1$ is contained in $\{\psi'(h\xi) \neq 0\} \subset \{\varepsilon' \leq h|\xi| \leq 2\varepsilon'\}$. Hence, $|D_x| \sim h^{-1}$ on the wavefront set of $L_1$.

Now, we redo the positive commutator argument with a cutoff $\chi(x)$ with $\chi(x) \equiv 1$ near $x = 0$ and $\chi(x) = 0$ away from $x = 0$ with $\chi^{1/2}$ still smooth. Let $B = f(x)\partial_x$, where $f(x) = \arctan(x)$, and

$$G = \int_0^T \langle \chi[D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1, B]u_{lo}, u_{lo}\rangle dt. \tag{4.4.2}$$

Then,

$$\begin{aligned} G &= \int_0^T \langle \chi(-2\langle x\rangle^{-2}\partial_x^2 - \tilde{V}_{p,n}' f(x) - \tilde{V}_1' f(x))u_{lo}, u_{lo}\rangle dt + \int_0^T \langle -\chi f''(x)\partial_x u_{lo}, u_{lo}\rangle dt \\ &\leq \int_0^T \langle \chi(-2\langle x\rangle^{-2}\partial_x^2 - \tilde{V}_{p,n}' f(x) - \tilde{V}_1' f(x))u_{lo}, u_{lo}\rangle dt + C_T||u_{lo}||_{H^{1/2}}^2 \end{aligned} \tag{4.4.3}$$

for a constant $C_T > 0$ dependent on $T$. The second line in (4.4.3) follows from the estimate (3.2.7). Recall that $\arctan(x)$ and $\tilde{V}_{p,n}'$ have opposite signs, so $-\chi \arctan(x)\tilde{V}_{p,n}' \geq 0$.

Additionally, $|\chi\tilde{V}' f(x)| \leq C$ for some constant $C$ independent of $p$ and $n$. Hence,

$$\int_0^T \langle \chi(-2\langle x\rangle^{-2}\partial_x^2 - \tilde{V}'_{p,n} f(x) - \chi\tilde{V}'_1 f(x))u_{lo}, u_{lo}\rangle dt \\ \geq \int_0^T \langle -2\chi\langle x\rangle^{-2}\partial_x^2 u_{lo}, u_{lo}\rangle dt - C_T||u_0||^2_{L^2} \tag{4.4.4}$$

for a constant $C_T$ independent of $p$ and $n$, but dependent on $T$. On the microsupport of $u_{lo}$, $|1/h| \lesssim |D_x|$. The Gårding inequality implies that there is some $C, C' > 0$ such that

$$\langle h^{-2}\chi u_{lo}, u_{lo}\rangle \leq -C\langle \chi\langle x\rangle^{-2}\partial_x^2 u_{lo}, u_{lo}\rangle + C'||u_{lo}||^2_{H^{1/2}(X)}. \tag{4.4.5}$$

Hence,

$$\begin{aligned} \int_0^T \langle h^{-2}\chi u_{lo}, u_{lo}\rangle dt &\leq -\int_0^T C\langle \chi\langle x\rangle^{-2}\partial_x^2 u_{lo}, u_{lo}\rangle dt + C'||u_{lo}||^2_{H^{1/2}(X)} \\ &\leq C\int_0^T \langle \chi(-2\langle x\rangle^{-2}\partial_x^2 - \tilde{V}'_{p,n} f(x) - \tilde{V}'_1 f(x))u_{lo}, u_{lo}\rangle dt + C_T||u_{lo}||^2_{H^{1/2}(X)} \\ &= C\int_0^T \langle \chi[D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1, B]u_{lo}, u_{lo}\rangle dt + C_T||u_0||^2_{H^{1/2}(X)}. \end{aligned}$$

The first line above follows from (4.4.5), the second line follows from (4.4.4), and the last line follows from (4.4.3). Rearranging gives,

$$\int_0^T \langle h^{-1}\chi u_{lo}, h^{-1}u_{lo}\rangle dt \leq C_T||u_0||^2_{H^{1/2}} + C\left|\int_0^T \langle \chi[D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1, B]u_{lo}, u_{lo}\rangle dt\right|.$$

Next we unpack the commutator in $G$ another way. Recall,

$$\begin{aligned} G &= \int_0^T \langle \chi(D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1)Bu_{lo}, u_{lo}\rangle dt - \int_0^T \langle \chi B(D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1)u_{lo}, u_{lo}\rangle dt \\ &:= G_1 + G_2. \end{aligned}$$

Integrating by parts with the $D_t$ and $D_x^2$ term in $G_1$ gives

$$\begin{aligned}|G_1| \leq& \left|\int_0^T \langle \chi B u_{lo}, (D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1)u_{lo}\rangle dt\right| + 2\left|\int_0^T \langle B u_{lo}, \chi' \partial_x u_{lo}\rangle dt\right| \\ &+ \left|\int_0^T \langle B u_{lo}, \chi'' u_{lo}\rangle dt\right| + \left|\langle B u_{lo}, u_{lo}\rangle|_0^T\right|.\end{aligned}$$

Integrating by parts with $B$ in $G_2$ gives

$$\begin{aligned}|G_2| \leq& \left|\int_0^T \langle (D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1)u_{lo}, (\chi f(x))' u_{lo}\rangle dt\right| \\ &+ \left|\int_0^T \langle \chi B u_{lo}, (D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1)u_{lo}\rangle dt\right|.\end{aligned}$$

Combining the estimates for $G_1$ and $G_2$ gives

$$\begin{aligned}|G| \leq& 2\left|\int_0^T \langle \chi B u_{lo}, (D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1)u_{lo}\rangle dt\right| + 2\left|\int_0^T \langle B u_{lo}, \chi' \partial_x u_{lo}\rangle dt\right| + \left|\langle B u_{lo}, u_{lo}\rangle|_0^T\right| \\ &+ \left|\int_0^T \langle B u_{lo}, \chi'' u_{lo}\rangle dt\right| + \left|\int_0^T \langle (D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1)u_{lo}, (\chi f(x))' u_{lo}\rangle dt\right|.\end{aligned}$$

Recall that $B = \arctan(x)\partial_x$. Since $\arctan(x)$ is bounded, $\chi'(x)$ is bounded, and there is perfect local smoothing in the $x$-direction by Lemma 4.3.42,

$$\left|\int_0^T \langle B u_{lo}, 2\chi' \partial_x u_{lo}\rangle dt\right| \leq C\int_0^T \int_{\mathbb{R}} |\langle x\rangle^{-3/2}\partial_x \tilde{u}|^2 dx dt \leq C_T ||u_0||_{H^{1/2}}^2. \tag{4.4.6}$$

Since $\chi'', \arctan(x)$ are bounded, using energy estimates and (4.4.6) gives

$$\left|\int_0^T \langle B u_{lo}, 2\chi' \partial_x u_{lo}\rangle dt\right| + \left|\int_0^T \int_{\mathbb{R}} \langle B u_{lo}, \chi'' u_{lo}\rangle dt\right| \leq C_T ||u_0||_{H^{1/2}}^2.$$

Next, recall that $(D_t + D_x^2 + \tilde{V})u_{lo} = h^{-1}L_1(\tilde{u})$. Let $\tilde{\psi}$ be a smooth, even, compactly supported bump function with $\tilde{\psi}(s) \equiv 1$ on $\operatorname{supp}(\psi)$.

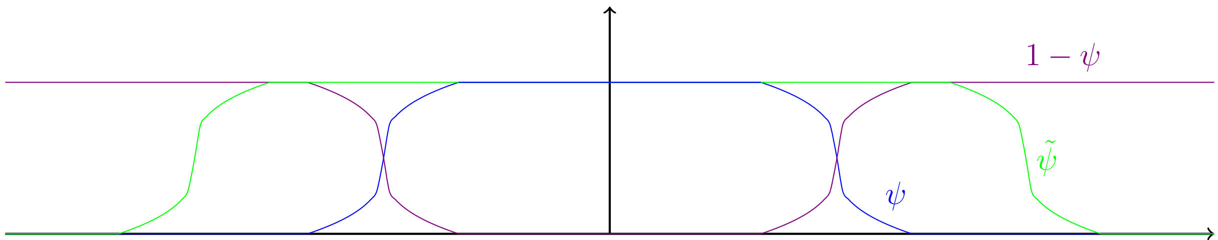


Figure 4.2: Comparing $\psi$, $1-\psi$ and $\tilde{\psi}$

The idea is that $\operatorname{supp}(\psi') \subseteq \operatorname{supp}(\psi) \subseteq \{\xi \in \mathbb{R} | \tilde{\psi}(\xi) \equiv 1\}$. This implies that

$$L_1\tilde{\psi}(hD_x) = L_1 + \mathcal{O}(h^\infty).$$

Let $\tilde{\chi}$ be a smooth compactly supported function such that $\tilde{\chi}(s) \equiv 1$ on $\operatorname{supp}(\chi)$. This construction is similar to $\tilde{\psi}$ so that $\operatorname{supp}(\chi') \subseteq \operatorname{supp}(\chi) \subseteq \{x \in \mathbb{R} | \tilde{\chi}(x) \equiv 1\}$.

Hence,

$$\begin{aligned}
&\left| \int_0^T \langle h^{-1} L_1 u_{lo}, (\chi \arctan(x))' u_{lo} \rangle dt \right| \\
&\qquad = \left| \int_0^T \langle h^{-1} L_1 \tilde{\psi} u_{lo}, (\chi \arctan(x))' u_{lo} \rangle dt \right| + \mathcal{O}(h^\infty) ||u_0||^2_{H^{1/2}} \\
&\qquad = \left| \int_0^T \langle h^{-1} \tilde{\chi} L_1 \tilde{\psi} u_{lo}, (\chi \arctan(x))' u_{lo} \rangle dt \right| + \mathcal{O}(h^\infty) ||u_0||^2_{H^{1/2}} \\
&\qquad \leq C \int_0^T ||h^{-1} \tilde{\chi} L_1 \tilde{\psi} u_{lo}||^2 dt + C \int_0^T ||(\chi \arctan(x))' u_{lo}||^2 dt + C_T ||u_0||^2_{H^{1/2}} \\
&\qquad \leq C \int_0^T ||h^{-1} \tilde{\chi} \tilde{\psi} \tilde{u}||^2 dt + C_T ||u_0||^2_{H^{1/2}}.
\end{aligned}$$

The estimate in the last line follows because $0 \leq \tilde{\psi}(1-\psi) \leq \tilde{\psi}$, $\arctan(x), \arctan'(x)$ are bounded, and that $\operatorname{supp}(\chi) \subset \{\tilde{\chi} \equiv 1\}$. Combining the estimates gives,

$$\begin{aligned}
\int_0^T \langle h^{-2}\chi u_{lo}, u_{lo}\rangle dt &\leq C_T ||u_0||^2_{H^{1/2}} + C\left|\int_0^T \langle \chi[D_t + D_x^2 + \tilde{V}_{p,n} + \tilde{V}_1, B]u_{lo}, u_{lo}\rangle dt\right| \\
&\leq C_T ||u_0||^2_{H^{1/2}} + C\left|\int_0^T \langle \chi B u_{lo}, h^{-1}L_1(\tilde{u})\rangle dt\right| + C\int_0^T ||h^{-1}\tilde{\chi}\tilde{\psi}\tilde{u}||^2 dt \\
&\leq C_T ||u_0||^2_{H^{1/2}} + C\left|\int_0^T \langle \chi B u_{lo}, h^{-1}\tilde{\chi}L_1\tilde{\psi}(\tilde{u})\rangle dt\right| + C\int_0^T ||h^{-1}\tilde{\chi}\tilde{\psi}\tilde{u}||^2 dt \\
&\leq C_T ||u_0||^2_{H^{1/2}} + C\int_0^T ||h^{-1}\tilde{\chi}\tilde{\psi}(\tilde{u})||^2 dt + C\int_0^T ||\chi B u_{lo}||^2 dt
\end{aligned}$$

for constants $C, C_T > 0$. Note that

$$\int_0^T ||\chi B u_{lo}||^2 dt \leq C_T ||u_0||^2_{H^{1/2}}$$

for some constant $C_T$ by Lemma 4.3.42 so that,

$$\int_0^T \langle h^{-2}\chi u_{lo}, u_{lo}\rangle dt \leq C_T ||u_0||^2_{H^{1/2}} + C\int_0^T ||h^{-1}\tilde{\chi}\tilde{\psi}\tilde{u}||^2 dt.$$

Since $u_{hi} = \psi\tilde{u}$, we have

$$||h^{-1}\chi^{1/2}u_{hi}||^2_{L^2} \leq ||h^{-1}\tilde{\chi}u_{hi}||^2_{L^2} \leq ||h^{-1}\tilde{\chi}\tilde{\psi}\tilde{u}||^2_{L^2} + \mathcal{O}(1)||u||^2_{L^2}.$$

That means

$$\int_0^T ||h^{-1}\chi^{1/2}u_{hi}||^2 dt \leq C\int_0^T ||h^{-1}\tilde{\chi}\tilde{\psi}\tilde{u}||^2 dt + C_T ||u_0||^2_{H^{1/2}}.$$

This implies that if

$$\int_0^T ||h^{-1}\tilde{\chi}\tilde{\psi}\tilde{u}||^2 dt \leq C\int_0^T ||h^{-m/(m+1)}\tilde{u}||^2_{L^2} dt + C_T ||u_0||^2_{H^{1/2}}, \tag{4.4.7}$$

then we can control the high and low frequency parts and get the necessary estimate.

### 4.4.2 High Frequency Estimate

We will use the $FF^*$ argument employed in [CW13] and [CM14]. For this section we will remove the tilde notation on $\chi, \psi, u, V$ and $V_1$ for convenience. We are considering functions $\chi(x)$ supported near $x = 0$ and $\psi(hD_x)$ micro-supported near 0. Let $F(t)$ be defined by

$$F(t)f(x) = \chi(x)\psi(hD_x)h^{-r}e^{-it(D_x^2+V+V_1)}f(x)$$

where $e^{-it(D_x^2+V+V_1)}$ is the Schrödinger propagator. We want to show that for some $r$ we have a mapping $F : L_x^2 \to L^2([0,T])L_x^2$, since then

$$||h^{r-1}F(t)u_0||_{L^2([0,T]);L_x^2} \leq C||h^{r-1}u_0||_{L^2}$$

gives (4.4.7) for $r = 1/m + 1$. We have such a mapping if and only if $FF^* : L^2([0,T])L_x^2 \to L^2([0,T])L_x^2$. Computing we get that

$$FF^*f(x,t) = \chi(x)\psi(hD_x)h^{-2r}\int_0^T e^{i(t-s)(D_x^2+V+V_1)}\psi(hD_x)\chi(x)f(x,s)ds$$

and need to show that $||FF^*f||_{L^2L^2} \leq C||f||_{L^2L^2}$. Next let $FF^*f(x,t) = \chi\psi(v_1 + v_2)$ where

$$v_1 = h^{-2r}\int_0^t e^{i(t-s)(D_x^2+V+V_1)}\psi(hD_x)\chi(x)f(x,s)ds,$$

and

$$v_2 = h^{-2r}\int_t^T e^{i(t-s)(D_x^2+V+V_1)}\psi(hD_x)\chi(x)f(x,s)ds,$$

so that

$$(D_t + D_x^2 + V + V_1)v_j = \pm ih^{-2r}\psi\chi f.$$

Then it is sufficient to estimate

$$||\chi\psi v_j||_{L^2L^2} \leq C||f||_{L^2L^2}.$$

Taking the Fourier transform in time and using Plancherel's theorem, we have that it is sufficient to estimate

$$||\chi\psi\hat{v}_j||_{L^2L^2} \leq C||\hat{f}||_{L^2L^2}.$$

This is the same as estimating

$$||\chi\psi h^{-2r}(\tau \pm i0 + D_x^2 + V + V_1)^{-1}\psi\chi||_{L_x^2\to L_x^2} \leq C. \tag{4.4.8}$$

for $C$ independent of $\tau$. Hence, we get the desired result if we can show (4.4.8). Let $-z = \tau h^2$, then (4.4.8) is equivalent to

$$||\chi(x)\psi(hD_x)(-z \pm i0 + (hD_x)^2 + h^2V + h^2V_1)^{-1}\psi(hD_x)\chi(x)||_{L^2\to L^2} \leq Ch^{-2(1-r)}. \tag{4.4.9}$$

For convenience, let

$$Q = (hD_x)^2 + h^2V + h^2V_1.$$

**Remark 4.4.10.** *From Section 3.3 we would expect the estimate to hold for $r = 1/(m+1)$ in general. Everything up until this point holds for any integer $m \geq 2$.*

Now, we will consider the case $m = 2$ and $r = 1/3$. If we can show for $h, \delta > 0$ sufficiently small that for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with compact support in $\{|(x,\xi)| \leq \delta)\}$

$$||(Q-z)\varphi^w u||_{L^2} \geq Ch^{4/3}||\varphi^w u||_{L^2},\ z \in [\tilde{M}-\delta, \tilde{M}+\delta] \tag{4.4.11}$$

for a constant $C > 0$, then we have (4.4.9). Instead we will prove

$$||(Q_1-z)\varphi^w u||_{L^2} \geq Ch^{4/3}||\varphi^w u||_{L^2} \tag{4.4.12}$$

for

$$Q_1 = (hD_x)^2 + h^2 V, \tag{4.4.13}$$

but this implies (4.4.11). Note that $V_1 = \mathcal{O}(1)$, $h^2 V_1 = \mathcal{O}(h^2)$. If (4.4.12) holds, then

$$||(Q-z)\varphi^w u||_{L^2} \geq ||(Q_1 - z)\varphi^w u||_{L^2} - \mathcal{O}(h^2)||\varphi^w u||_{L^2} \geq Ch^{4/3}||\varphi^w u||_{L^2}$$

showing (4.4.11). We will get (4.4.12) by computing the commutator, $|\langle [Q_1 - z, a^w]v, v\rangle|$ for an appropriate symbol $a$. This is where we will have to use Lemma 2.3.24 and specifically equation (2.3.25). In the following sections we define the symbol $a$ to give the right properties and then calculate the terms in the commutator using Lemma 2.3.24.

**Remark 4.4.14.** *Away from the microlocal resolvent estimate in* (4.4.11), $Q = \tilde{q}^w$ *for a non-trapping symbol* $\tilde{q}$. *The gluing technique from* [Chr18] *combined with* (4.4.11) *give Theorem 4.2.2*

### 4.4.3 Defining the Commutant

For this section assume that $|p| \geq |n| > 0$ and $h^{-2} = p^2 + n^2$. The situation where $|n| \geq |p| > 0$ will follow similarly. We will also introduce a second parameter $\tilde{h}$ such that $\tilde{h} \geq h$. When $n = 0$ we then only have a single term affecting the potential so we can use the estimates from [CW13]. Furthermore, we will consider case the where $m = 2$ so that $-V'(x) = E_1 x - E_2 x^2 + E_3 x^3$. We will give what $E_1$, $E_2$, and $E_3$ are later. We will also be using the change of variables,

$$X = \left(\frac{\tilde{h}}{h}\right)^{1/3} x, \;\; \Xi = \left(\frac{\tilde{h}}{h}\right)^{2/3} \xi.$$

We want to estimate the symbol $\{q, a\}$ where $q = \xi^2 + h^2 V(x)$ and

$$a = \Lambda\left(\left(\frac{\tilde{h}}{h}\right)^{1/3} x\right)\Lambda\left(\left(\frac{\tilde{h}}{h}\right)^{2/3} \xi\right)\chi_2(x)\chi_2(\xi) = \Lambda(X)\Lambda(\Xi)\chi_2(x)\chi_2(\xi) \tag{4.4.15}$$

where

$$\Lambda(s) = \int_0^s \lambda(x)dx$$

such that

$$\lambda(x) = \begin{cases} 1, & |x| < \varepsilon_\lambda \\ \frac{C_{\varepsilon_\lambda}}{|x|^{3/2}}, & |x| > 2\varepsilon_\lambda \end{cases}$$

$\lambda(x)$ is even, $\lambda'(x) \leq 0$ for $x \geq 0$, $\lambda'(x) \geq 0$ for $x \leq 0$, $C_{\varepsilon_\lambda}$ is chosen so that $\lambda$ is smooth, and $0 < \chi_2(x) \leq 1$ is smooth cutoff near 0. Note $\Lambda(x)$ and all of its derivatives are bounded. Furthermore,

$$|\Lambda'''(x)| \leq C''_{\varepsilon_\lambda} \Lambda'(x)$$

for some constant $C''_{\varepsilon_\lambda} > 0$ due to the construction of $\lambda$. Additionally, $\Lambda'''(x) = 0$ for $|x| < \varepsilon_\lambda$.

#### 4.4.4 Estimating the first term in the Commutator

Let $\mathcal{B}$ denote the blowdown map

$$(x, \xi) = \mathcal{B}(X, \Xi) = \left( \left(\frac{h}{\tilde{h}}\right)^{1/3} X, \left(\frac{h}{\tilde{h}}\right)^{2/3} \Xi \right)$$

and let $T_{h,\tilde{h}}$ denote the unitary operator

$$T_{h,\tilde{h}} u(X) = \left(\frac{h}{\tilde{h}}\right)^{1/6} u\left( \left(\frac{h}{\tilde{h}}\right)^{1/3} X \right)$$

so that if $g \in \mathcal{S}^{k,m,\tilde{m}}_{1/3,2/3}$, then

$$
\begin{aligned}
&\langle \operatorname{Op}_h(g(x,\xi))u(x), u(x)\rangle \\
&= \int \frac{1}{2\pi h}\int\int e^{\frac{i}{h}(x-y)\xi} g\Big(\frac{x+y}{2},\xi\Big)u(y)dyd\xi\bar{u}(x)dx \\
&= \int \frac{1}{2\pi h}\int\int e^{\frac{i}{h}((\frac{h}{\tilde{h}})^{1/3}X-y)\xi} g\bigg(\frac{(\frac{h}{\tilde{h}})^{1/3}X+y}{2},\xi\bigg)u(y)dyd\xi\bar{u}\bigg(\Big(\frac{h}{\tilde{h}}\Big)^{1/3}X\bigg)\Big(\frac{h}{\tilde{h}}\Big)^{1/3}dX \\
&= \int \frac{\tilde{h}^{-1/6}}{2\pi h^{5/6}}\int\int e^{\frac{i}{\tilde{h}}(X-Y)\Xi} g\bigg(\frac{(\frac{h}{\tilde{h}})^{1/3}(X+Y)}{2},\Big(\frac{h}{\tilde{h}}\Big)^{2/3}\Xi\bigg)u\bigg(\Big(\frac{h}{\tilde{h}}\Big)^{1/3}Y\bigg)\frac{h}{\tilde{h}}dYd\Xi\overline{T_{h,\tilde{h}}u}(X)dX \\
&= \int \frac{1}{2\pi \tilde{h}}\int\int e^{\frac{i}{\tilde{h}}(X-Y)\Xi} g\bigg(\frac{(\frac{h}{\tilde{h}})^{1/3}(X+Y)}{2},\Big(\frac{h}{\tilde{h}}\Big)^{2/3}\Xi\bigg)T_{h,\tilde{h}}u(Y)dYd\Xi\overline{T_{h,\tilde{h}}u}(X)dX \\
&= \int \frac{1}{2\pi \tilde{h}}\int\int e^{\frac{i}{\tilde{h}}(X-Y)\Xi} (g\circ\mathcal{B})\Big(\frac{(X+Y)}{2},\Xi\Big)T_{h,\tilde{h}}u(Y)dYd\Xi\overline{T_{h,\tilde{h}}u}(X)dX \\
&= \langle \operatorname{Op}_{\tilde{h}}(g\circ\mathcal{B})(X,\Xi)(T_{h,\tilde{h}}u)(X), T_{h,\tilde{h}}u(X)\rangle.
\end{aligned}
$$

The first term in $[Q,A]$ is given by $h\operatorname{Op}^w_h\{q,a\}$ since we are using the $h$-Weyl quantization. The estimate we want in the end is of the form

$$
h\langle \operatorname{Op}^w_h(\{q,a\})u,u\rangle \geq Ch^{4/3}\tilde{h}||u||^2_{L^2}.
$$

After estimating this term we will also have to deal with the other terms from the commutator.

To estimate this first term we will use the following process of estimates,

$$
\begin{aligned}
h\langle \operatorname{Op}^w_h(\{q,a\})(x,\xi)u(x),u(x)\rangle &= h\langle \operatorname{Op}^w_{\tilde{h}}(\{q,a\}\circ\mathcal{B})(X,\Xi)T_{h,\tilde{h}}u(X),T_{h,\tilde{h}}u(X)\rangle \\
&= h(\frac{h}{\tilde{h}})^{1/3}\langle \operatorname{Op}^w_{\tilde{h}}(g_1)(X,\Xi)T_{h,\tilde{h}}u(X),T_{h,\tilde{h}}u(X)\rangle - \text{Error Terms} \qquad (4.4.16)\\
&\geq h^{4/3}\tilde{h}||u||^2_{L^2} - \text{Error Terms} \qquad (4.4.17)\\
&\geq Ch^{4/3}\tilde{h}||u||_{L^2}s^2.
\end{aligned}
$$

**Remark 4.4.18.** *These estimates are not exact and avoid details that will be cleared up in the calculations. However, it gives the idea of how we will estimate the first term in the* *commutator.*

The estimate in (4.4.16) will come from calculating $\{q,a\}\circ\mathcal{B}$ and dealing with terms supported away from the critical point. The main work of the section is the estimate in (4.4.17). This will require us to carefully calculate a lower bound for $g_1$. Once, we get the correct lower bound we can use the process from [CW13] to get the estimate.

To start we will compute the Poisson bracket,

$$\begin{aligned} h\{q,a\}(x,\xi) &= h\Big(2\xi\Lambda\big(\Xi\big)\big[h^{-1/3}\tilde{h}^{-1/3}\lambda\big(X\big)\big] - h^2V'(x)\Lambda\big(X\big)\big[h^{-2/3}\tilde{h}^{2/3}\lambda\big(\Xi\big)\big]\Big)\chi_2(x)\chi_2(\xi)+r \\ &= \Bigg(h^{2/3}\tilde{h}^{1/3}2\xi\Lambda\big(\Xi\big)\lambda\big(X\big) - h^{1/3}\tilde{h}^{2/3}h^2V'(x)\Lambda\big(X\big)\lambda\big(\Xi\big)\Bigg)\chi_2(x)\chi_2(\xi)+r \\ &= g\chi_2(x)\chi_2(\xi)+r. \end{aligned} \tag{4.4.19}$$

The $r$ term contains the derivatives of $\chi_2$ which are supported away from the critical point. Recall that $V'_{p,n}(x) = -4p^2(x-\varepsilon)^3 - 4n^2(x+\varepsilon)^3$, so

$$-V'_{p,n}(x)(x-x_0) = (4p^2(x-\varepsilon)^3 + 4n^2(x+\varepsilon)^3)(x-x_0) \geq 0$$

from Proposition 4.3.15. This implies that $-V'(x)\Lambda(X)\geq 0$, however we will need to show better bounds. Now, that we calculated $\{q,a\}$ and defined $g$ we want to prove the following estimate,

**Lemma 4.4.20.** *Let* $h^{4/3}\tilde{h}^{-1/3}g_1(X,\Xi;h) = (g\circ\mathcal{B})(X,\Xi;h) = g(x,\xi;h)$ *where* $g$ *is defined in* (4.4.19)*. Then,*

$$g_1(X,\Xi;h) \geq \begin{cases} C_g(\Xi^2+X^4), & |X|\leq\varepsilon_\lambda, |\Xi|\leq\varepsilon_\lambda \\ C_g, & \textit{otherwise} \end{cases} \tag{4.4.21}$$

*where* $C_g>0$ *independent of* $h$ *and* $\tilde{h}$*.*

Now to prove the lemma we have to look at what happens to $g$ in the different cases for

$\Lambda\left(\left(\frac{\tilde{h}}{h}\right)^{2/3}\xi\right)$ and $\Lambda\left(\left(\frac{\tilde{h}}{h}\right)^{1/3}x\right)$. Hence, we have 4 different cases we need to look at.

1. $|x| \leq \varepsilon_\lambda\left(\frac{h}{\tilde{h}}\right)^{1/3}$ and $|\xi| \leq \varepsilon_\lambda\left(\frac{h}{\tilde{h}}\right)^{2/3}$
2. $|x| \leq \varepsilon_\lambda\left(\frac{h}{\tilde{h}}\right)^{1/3}$ and $|\xi| > \varepsilon_\lambda\left(\frac{h}{\tilde{h}}\right)^{2/3}$
3. $|x| > \varepsilon_\lambda\left(\frac{h}{\tilde{h}}\right)^{1/3}$ and $|\xi| \leq \varepsilon_\lambda\left(\frac{h}{\tilde{h}}\right)^{2/3}$
4. $|x| > \varepsilon_\lambda\left(\frac{h}{\tilde{h}}\right)^{1/3}$ and $|\xi| > \varepsilon_\lambda\left(\frac{h}{\tilde{h}}\right)^{2/3}$

After the blowdown map is applied this will give use the following regions for $g \circ \mathcal{B}$

1. $|X| \leq \varepsilon_\lambda$ and $|\Xi| \leq \varepsilon_\lambda$
2. $|X| \leq \varepsilon_\lambda$ and $|\Xi| > \varepsilon_\lambda$
3. $|X| > \varepsilon_\lambda$ and $|\Xi| \leq \varepsilon_\lambda$
4. $|X| > \varepsilon_\lambda$ and $|\Xi| > \varepsilon_\lambda$

This is why the blowdown map is used. It allows us to switch from $h$-dependent cases to non $h$-dependent cases. To handle these cases we will need to provide estimates on $-V'(x)$.

*Estimating $-V'(x)$*

In this section we will need to find the critical point for $V'(x)$. The critical point $x_{p,n}$ is given by,

$$
\begin{aligned}
0 &= -4p^2(x_{p,n}-\varepsilon)^3 - 4n^2(x_{p,n}+\varepsilon)^3 \\
n^2(x_{p,n}+\varepsilon)^3 &= -p^2(x_{p,n}-\varepsilon)^3 \\
\sqrt[3]{\frac{-n^2}{p^2}} &= \frac{(x_{p,n}-\varepsilon)}{(x_{p,n}+\varepsilon)} \\
\left(1-\sqrt[3]{\frac{-n^2}{p^2}}\right)x_{p,n} &= \varepsilon\left(1+\sqrt[3]{\frac{-n^2}{p^2}}\right) \\
x_{p,n} &= \varepsilon\left(\frac{1+\sqrt[3]{\frac{-n^2}{p^2}}}{1-\sqrt[3]{\frac{-n^2}{p^2}}}\right) \\
x_{p,n} &= \varepsilon\left(\frac{p^{2/3}-n^{2/3}}{p^{2/3}+n^{2/3}}\right)
\end{aligned}
$$

Before shifting the function $-V(x)$ we have,

$$E_1 = \frac{2V''(x_{p,n})}{2!} = 48\varepsilon^2\frac{n^2p^{4/3}+p^2n^{4/3}}{(p^{2/3}+n^{2/3})^2} \tag{4.4.22}$$

$$-E_2 = \frac{3V'''(x_{p,n})}{3!} = -24\varepsilon\frac{-n^2p^{2/3}+p^2n^{2/3}}{p^{2/3}+n^{2/3}} \tag{4.4.23}$$

$$E_3 = \frac{4V''''(x_{p,n})}{4!} = 4(p^2+n^2). \tag{4.4.24}$$

The choice of $E_2$ is so that $E_2 \geq 0$ when $|p| \geq |n| > 0$. We can use $E_1, E_2$ and $E_3$ and the fact that $-\tilde{V}'(0) = 0$ to define $-\tilde{V}'(x)$. We are also going to drop the $p, n$ and tilde notation for the shifted $V$ function for convenience. This gives

$$-V'(x) = E_1x - E_2x^2 + E_3x^3. \tag{4.4.25}$$

Recall that we are using $h^{-2} = p^2 + n^2$ and that $E_2 \geq 0$. We will want to prove the following lemma,

**Lemma 4.4.26.** *Let* $|x| \le 2\varepsilon$ *and* $-V'(x) = E_1x - E_2x^2 + E_3x^3$ *where* $E_1$*,* $E_2$ *and* $E_3$ *are defined in* (4.4.22), (4.4.23), *and* (4.4.24) *respectively. We have the following estimate,*

$$-h^2V'(x)\Lambda(X) \ge h^2\left(\frac{1}{10}E_1x\Lambda(X) + \frac{1}{10}E_3x^3\Lambda(X)\right)$$
$$= \frac{2}{5}\left(\frac{E_1}{E_3}x\Lambda(X) + x^3\Lambda(X)\right). \tag{4.4.27}$$

**Remark 4.4.28.** *The choice of* $\frac{1}{10}$ *is not optimal, but is sufficient for the estimates needed in this thesis.*

*Proof of Lemma 4.4.26.* We want to show that

$$-V'(x)\Lambda(X) = (E_1 - E_2x + E_3x^2)x\Lambda(X) \ge kE_1x\Lambda(X) + \tilde{k}E_3x^3\Lambda(X)$$

for some constants $k, \tilde{k} > 0$. This will hold if

$$\tilde{f}(x) := ((1-k)E_1 - E_2x + E_3(1-\tilde{k})x^2) \ge 0,$$

since $x\Lambda(X) \ge 0$. Now $\tilde{f}(x)$ is a quadratic function in $x$, so we will show that for $k, \tilde{k}$ sufficiently small that the minimum of $\tilde{f}(x) \ge 0$ for all $|p| \ge |n| > 0$. First note that the minimum occurs at $E_2/((1-\tilde{k})2E_3)$. The minimum is then,

$$\begin{aligned}\tilde{f}\left(\frac{E_2}{(1-\tilde{k})2E_3}\right) &= (1-k)E_1 - \frac{E_2^2}{4(1-\tilde{k})E_3}\\ &= (1-k)48\varepsilon^2\frac{n^2p^{4/3}+p^2n^{4/3}}{(p^{2/3}+n^{2/3})^2} - \frac{24^2\varepsilon^2(-n^2p^{2/3}+p^2n^{2/3})^2}{16(1-\tilde{k})(p^{2/3}+n^{2/3})^2(p^2+n^2)}\\ &= \frac{24\varepsilon^2}{(p^{2/3}+n^{2/3})^2}\left(2(1-k)(n^2p^{4/3}+p^2n^{4/3}) - \left(\frac{3(p^2n^{2/3}-n^2p^{2/3})^2}{2(1-\tilde{k})(p^2+n^2)}\right)\right).\end{aligned} \tag{4.4.29}$$

Let $d = \frac{n}{p}$ or equivalently $n = dp$. Since $|p| \ge |n| > 0$, we have that $-1 \le d \le 1$. Substituting

$n = dp$ into (4.4.29) to get rid of $n$ gives

$$f\left(\frac{E_2}{(1-\tilde{k})2E_3}\right) = \frac{24\varepsilon}{(1+d^{2/3})^2p^{4/3}}\left(2(1-k)(d^{4/3}+d^2)p^{10/3} - \left(\frac{3(d^{2/3}-d^2)^2p^{16/3}}{2(1-\tilde{k})(1+d^2)p^2}\right)\right)$$
$$= \frac{24\varepsilon d^{4/3}p^2}{(1+d^{2/3})^2}\left(2(1-k)(1+d^{2/3}) - \left(\frac{3}{2(1-\tilde{k})}\right)\left(\frac{(1-d^{4/3})^2}{(1+d^2)}\right)\right). \quad (4.4.30)$$

Notice that

$$\frac{24\varepsilon d^{4/3}p^2}{(1+d^{2/3})^2} \geq 0$$

for all $d$. This implies that

$$\left(2(1-k)(1+d^{2/3}) - \left(\frac{3}{2(1-\tilde{k})}\right)\left(\frac{(1-d^{4/3})^2}{(1+d^2)}\right)\right) \geq 0$$

for all $-1 \leq d \leq 1$, if

$$2(1-k) - \frac{3}{2(1-\tilde{k})} \geq 0.$$

Taking $k = \tilde{k} = 1/10$ gives $2(1-k) - \frac{3}{2(1-\tilde{k})} = 18/10 - 30/18 = 2/15 > 0$. Hence, $\tilde{f}(x) \geq 0$ for all $x$ and $|p| \geq |n| > 0$. This proves Lemma 4.4.26. □

Next we want to estimate $g \circ \mathcal{B}$ using the lemma. This idea follows the "blow-down" map used in [CW13] which is described in section 3. We will do this by breaking up into the 4 cases.

**Remark 4.4.31.** *In [CW13] and [CM14], estimates are done initially in the $X, \Xi$ variables and then $\mathcal{B}^{-1}$ is used to get back to the initial symbol in the $x$ and $\xi$ variables. We will avoid this method, since it was easier to get inequalities initially in the $x$ and $\xi$ variables and then use $\mathcal{B}$.*

*Case 1*

Let $|x| \leq \varepsilon_\lambda \left(\frac{h}{\tilde{h}}\right)^{1/3}$ and $|\xi| \leq \varepsilon_\lambda \left(\frac{h}{\tilde{h}}\right)^{2/3}$. In this case

$$\begin{aligned}
g &= h^{2/3}\tilde{h}^{1/3} 2\xi \Lambda\left(\left(\frac{\tilde{h}}{h}\right)^{2/3}\xi\right)\lambda\left(\left(\frac{\tilde{h}}{h}\right)^{1/3}x\right) - h^{1/3}\tilde{h}^{2/3}h^2 V'(x)\Lambda\left(\left(\frac{\tilde{h}}{h}\right)^{1/3}x\right)\lambda\left(\left(\frac{\tilde{h}}{h}\right)^{2/3}\xi\right) \\
&= h^{2/3}\tilde{h}^{1/3} 2\xi \left(\frac{\tilde{h}}{h}\right)^{2/3}\xi - h^{1/3}\tilde{h}^{2/3}h^2 V'(x)\left(\frac{\tilde{h}}{h}\right)^{1/3}x \\
&\geq \tilde{h}\left(2\xi^2 + \frac{2E_1}{5E_3}x^2 + \frac{2}{5}x^4\right)
\end{aligned}$$

where we use (4.4.26) to get the last inequality. After applying the blowdown map we get,

$$\begin{aligned}
g \circ \mathcal{B}\left(X, \Xi\right) &\geq 2h^{4/3}\tilde{h}^{-1/3}\Xi^2 + \frac{2E_1}{5E_3}h^{2/3}\tilde{h}^{1/3}X^2 + \frac{2}{5}h^{4/3}\tilde{h}^{-1/3}X^4 \\
&\geq h^{4/3}\tilde{h}^{-1/3}\left(2\Xi^2 + \frac{2}{5}X^4\right) \qquad (4.4.32)
\end{aligned}$$

for $|X| \leq \varepsilon_\lambda$ and $|\Xi| \leq \varepsilon_\lambda$.

*Case 2*

When $|x| \leq \varepsilon_\lambda \left(\frac{h}{\tilde{h}}\right)^{1/3}$ and $|\xi| > \varepsilon_\lambda \left(\frac{h}{\tilde{h}}\right)^{2/3}$,

$$g \geq 2h^{2/3}\tilde{h}^{1/3}\left|\xi \Lambda\left(\left(\frac{h}{\tilde{h}}\right)^{2/3}\xi\right)\right| \geq 2h^{2/3}\tilde{h}^{1/3}\left(\varepsilon_\lambda \frac{h^{2/3}}{\tilde{h}^{2/3}}\right)\left|\Lambda(\Xi)\right| \geq 2\varepsilon_\lambda^2 h^{4/3}\tilde{h}^{-1/3}.$$

After applying the blowdown map, we get

$$g \circ \mathcal{B}\left(X, \Xi\right) \geq 2\varepsilon_\lambda^2 h^{4/3}\tilde{h}^{-1/3} \qquad (4.4.33)$$

for $|X| \leq \varepsilon_\lambda$ and $|\Xi| > \varepsilon_\lambda$.

*Case 3*

When $|x| \geq \varepsilon_\lambda \left(\frac{h}{\tilde{h}}\right)^{1/3}$ and $|\xi| \leq \varepsilon_\lambda \left(\frac{h}{\tilde{h}}\right)^{2/3}$,

$$g \geq h^{1/3}\tilde{h}^{2/3}(h^2 V'(x))\Lambda\Big(\Big(\frac{\tilde{h}}{h}\Big)^{1/3} x\Big).$$

By (4.4.26),

$$g \geq h^{1/3}\tilde{h}^{2/3}\Big(\frac{2E_1}{5E_3}x + \frac{2}{5}x^3\Big)\Lambda\Big(\Big(\frac{\tilde{h}}{h}\Big)^{1/3} x\Big). \tag{4.4.34}$$

Using the bounds on $|\Lambda(X)|$, $|x|$ and the fact $x$ and $\Lambda(X)$ have the same signs we get that

$$g \geq \frac{2}{5}h^{4/3}\tilde{h}^{-1/3}\varepsilon_\lambda^3\Big|\Lambda\Big(\Big(\frac{\tilde{h}}{h}\Big)^{1/3} x\Big)\Big| \geq \frac{2}{5}\varepsilon_\lambda^4 h^{4/3}\tilde{h}^{-1/3}.$$

After applying the blowdown map, we get

$$g \circ \mathcal{B}\,(X, \Xi) \geq \frac{2}{5}\varepsilon_\lambda^4 h^{4/3}\tilde{h}^{-1/3} \tag{4.4.35}$$

for $|X| \geq \varepsilon_\lambda$ and $|\Xi| \leq \varepsilon_\lambda$.

*Case 4*

When $|x| \geq \varepsilon_\lambda \left(\frac{h}{\tilde{h}}\right)^{1/3}$ and $|\xi| > \varepsilon_\lambda \left(\frac{h}{\tilde{h}}\right)^{2/3}$,

$$\begin{aligned} g &= \Big(2\xi h^{2/3}\tilde{h}^{1/3}\xi\Lambda\Big(\big(\frac{\tilde{h}}{h}\big)^{2/3}\xi\Big)\lambda\Big(\big(\frac{\tilde{h}}{h}\big)^{1/3}x\Big) - \tilde{h}^{2/3}h^{1/3}(h^2V'(x))\Lambda\Big(\big(\frac{\tilde{h}}{h}\big)^{1/3}x\Big)\lambda\Big(\big(\frac{\tilde{h}}{h}\big)^{2/3}\xi\Big)\Big) \\ &\geq \Big(\frac{2h^{2/3}\tilde{h}^{1/3}C_{\varepsilon_\lambda}\varepsilon_\lambda|\xi|}{|\big(\frac{\tilde{h}}{h}\big)^{1/3}x|^{3/2}} + \frac{\tilde{h}^{2/3}h^{1/3}|h^2V'(x)|C_{\varepsilon_\lambda}\varepsilon_\lambda}{|\big(\frac{\tilde{h}}{h}\big)^{2/3}\xi|^{3/2}}\Big). \end{aligned}$$

Using Lemma 4.4.26 gives

$$g \circ \mathcal{B}\,(X, \Xi) \geq \left( \frac{2h^{4/3}\tilde{h}^{-1/3} C_{\varepsilon_\lambda} \varepsilon_\lambda |\Xi|}{|X|^{3/2}} + \frac{\frac{2C_{\varepsilon_\lambda}\varepsilon_\lambda}{5} h^{4/3}\tilde{h}^{-1/3}|X|^3}{|\Xi|^{3/2}} \right). \tag{4.4.36}$$

First, consider the case when $\frac{|\Xi|}{|X|^{3/2}} \geq 1$. In this case

$$\begin{aligned} g \circ \mathcal{B}\,(X, \Xi) &\geq \left( \frac{2h^{4/3}\tilde{h}^{-1/3} C_{\varepsilon_\lambda} \varepsilon_\lambda |\Xi|}{|X|^{3/2}} + \frac{\frac{2C_{\varepsilon_\lambda}\varepsilon_\lambda}{5} h^{4/3}\tilde{h}^{-1/3}|X|^3}{|\Xi|^{3/2}} \right) \\ &\geq 2C_{\varepsilon_\lambda}\varepsilon_\lambda h^{4/3}\tilde{h}^{-1/3}. \end{aligned}$$

Now, consider the case when $\frac{|X|^{3/2}}{|\Xi|} \geq 1$. In this case

$$\begin{aligned} g \circ \mathcal{B}\,(X, \Xi) &\geq \left( \frac{2h^{4/3}\tilde{h}^{-1/3} C_{\varepsilon_\lambda} \varepsilon_\lambda |\Xi|}{|X|^{3/2}} + \frac{\frac{2C_{\varepsilon_\lambda}\varepsilon_\lambda}{5} h^{4/3}\tilde{h}^{-1/3}|X|^3}{|\Xi|^{3/2}} \right) \\ &\geq \frac{2C_{\varepsilon_\lambda}\varepsilon_\lambda}{5} h^{4/3}\tilde{h}^{-1/3}. \end{aligned} \tag{4.4.37}$$

Using both estimates and applying the blowdown map we get,

$$g \circ \mathcal{B}\,(X, \Xi) \geq C_4 h^{4/3}\tilde{h}^{-1/3}$$

for $|X| \geq \varepsilon_\lambda$ and $|\Xi| > \varepsilon_\lambda$, where $C_4$ is a positive constant independent of $h$.

*Combining Estimates*

Now, define $g_1$ such that $g(x, \xi) = (g \circ \mathcal{B}\,)(X, \Xi) = h^{4/3}\tilde{h}^{-1/3} g_1(X, \Xi)$. Then, combining the estimates from (4.4.32), (4.4.33), (4.4.35), and (4.4.37) we get

$$g_1(X, \Xi; h) \geq \begin{cases} C_g(\Xi^2 + X^4), & |X| \leq \varepsilon_\lambda \text{ and } |\Xi| \leq \varepsilon_\lambda \\ C_g, & \text{else} \end{cases} \tag{4.4.38}$$

for a positive $C_g > 0$ independent of $h$ and $\tilde{h}$. Now, that we have the bounds lower bound of $g_1$ we can use Lemma 2.5 and Lemma 2.6 from [CW13].

We will need the following lemma for the final estimate,

**Lemma 4.4.39.** *Let $\tilde{r} = \mathcal{O}_{\mathcal{S}_{1/3,2/3}}(\tilde{h}^{5/3})$. For $\tilde{h} > 0$ sufficiently small, there exists $c > 0$ such that*

$$\langle \mathrm{Op}\,_{\tilde{h}}^{w}(g_1(1+\tilde{r}))u, u\rangle \geq c\tilde{h}^{4/3}||u||_{L^2}^2$$

*uniformly as $h \downarrow 0$.*

The addition of the $\tilde{r}$ term will be needed to control the third order term.

*Proof.* Note that $\tilde{r} = \mathcal{O}_{\mathcal{S}_{1/3,2/3}}(\tilde{h}^{5/3})$, so take $\tilde{h}$ sufficiently small so that $1+\tilde{r} \geq c_r > 0$ for some constant $c_r$. Using that $1+\tilde{r} \geq c_r > 0$ and the estimate in (4.4.38), gives that $g_1(1+\tilde{r})$ is elliptic when $|X| > \varepsilon_\lambda$ or $|\Xi| > \varepsilon_\lambda$. This implies that there is a constant $C > 0$ independent of $\tilde{h} > 0$ such that if $\langle Op_{\tilde{h}}^{w}(g_1(1+\tilde{r}))u, u\rangle \leq C||u||_{L^2}^2$, $u$ has semiclassical wavefront set contained in the set $S = \{(X, \Xi) : |X| \leq \varepsilon_\lambda/2 \text{ and } |\Xi| \leq \varepsilon_\lambda/2\}$. On $S$, $g_1(1+\tilde{r}) = (\Xi^2 + X^4)K_g^2$ for a strictly positive symbol $K_g$. Note that the Weyl quantization has the convenient feature that

$$\langle \mathrm{Op}\,_{\tilde{h}}^{w}((\Xi^2 + X^4)K_g^2)u, u\rangle = \langle \mathrm{Op}\,_{\tilde{h}}^{w}(K_g)^*(\tilde{h}^2 D_X^2 + X^4)\mathrm{Op}\,_{\tilde{h}}^{w}(K_g)u, u\rangle + \mathcal{O}(\tilde{h}^2).$$

Additionally, as shown in Lemma A.2 of [CW13]

$$\langle (\tilde{h}^2 D_x^2 + X^4)u, u\rangle \geq \tilde{h}^{4/3}||u||_{L^2}^2.$$

Suppose $u$ has semiclassical wavefront set contained in the set $S = \{(X, \Xi) : |X| \leq \varepsilon_\lambda/2 \text{ and } |\Xi| \leq \varepsilon_\lambda/2\}$. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be a cutoff function such that $\varphi \equiv 1$ on $S$ and

$\varphi = 0$ when $|X| > \varepsilon_\lambda$ or $|\Xi| > \varepsilon_\lambda$. Then,

$$
\begin{aligned}
&\langle \mathrm{Op}\,^w_{\tilde h}(g_1(1+\tilde r))u, u\rangle \\
&\quad = \langle \mathrm{Op}\,^w_{\tilde h}(g_1(1+\tilde r))\varphi^w u, \varphi^w u\rangle + \langle Op^w_{\tilde h}(g_1(1+\tilde r))(1-\varphi)^w u, (1-\varphi)^w u\rangle \\
&\qquad + \langle \mathrm{Op}\,^w_{\tilde h}(g_1(1+\tilde r))\varphi^w u, (1-\varphi)^w u\rangle + \langle \mathrm{Op}\,^w_{\tilde h}(g_1(1+\tilde r))(1-\varphi)^w u, \varphi^w u\rangle \\
&\quad = \langle \mathrm{Op}\,^w_{\tilde h}(K_g)^*(\tilde h^2 D_X^2 + X^4)\mathrm{Op}\,^w_{\tilde h}(K_g)\varphi^w u, \varphi^w u\rangle + \mathcal{O}(\tilde h^2)||u||^2_{L^2} + \mathcal{O}(\tilde h^\infty)||u||^2_{L^2} \\
&\quad = \langle (\tilde h^2 D_X^2 + X^4)\mathrm{Op}\,^w_{\tilde h}(K_g)\varphi^w u, \mathrm{Op}\,^w_{\tilde h}(K_g)\varphi^w u\rangle + \mathcal{O}(\tilde h^2)||u||^2_{L^2} + \mathcal{O}(\tilde h^\infty)||u||^2_{L^2} \\
&\quad \geq \tilde h^{4/3} c_1 ||\mathrm{Op}\,^w_{\tilde h}(K_g)\varphi^w u||^2_{L^2} + \mathcal{O}(\tilde h^2)||u||^2_{L^2} + \mathcal{O}(\tilde h^\infty)||u||^2_{L^2} \\
&\quad \geq \tilde h^{4/3} c_1' ||\varphi^w u||^2_{L^2} + \mathcal{O}(\tilde h^2)||u||^2_{L^2} + \mathcal{O}(\tilde h^\infty)||u||^2_{L^2} \\
&\quad \geq \tilde h^{4/3} c_1'' ||u||^2_{L^2} + \mathcal{O}(\tilde h^2)||u||^2_{L^2} + \mathcal{O}(\tilde h^\infty)||u||^2_{L^2}
\end{aligned}
$$

for strictly positive constants $c_1, c_1'$ and $c_1''$. We use the fact that $u$ has semiclassical wavefront set contained in $S$ to go from lines two and three to line four. We use the fact that $K_g$ is strictly positive to go from line five to six. Additionally, we use that $u$ has semiclassical wavefront set contained in $S$ and $\varphi \equiv 1$ on S to go from line six to seven. So, taking $\tilde h$ sufficiently small gives the desired result.

□

### 4.4.5 Estimating the third order term in the commutant

Due to using the Weyl calculus we only have odd ordered derivatives in the expansion of the commutant. Furthermore, there are no mixed derivative terms since $\partial_x\partial_\xi q = 0$. Additionally $\partial_\xi^k q = 0$ for $k \geq 3$ and due to the construction of $V$, $\partial_x^k q = 0$ for $k \geq 4$. This means that the only additional term we need to estimate is the third order term given by,

$$
h^3 \partial_x^3 q \partial_\xi^3 a = h^3 \left( \left(\frac{\tilde h}{h}\right)^2 h^2 V'''(x) \Lambda'''\Big( \big(\frac{\tilde h}{h}\big)^{2/3} \xi \Big) \Lambda\Big( \big(\frac{\tilde h}{h}\big)^{1/3} x \Big) \chi_2(x)\chi_2(\xi) + r_2 \right)
$$

where $r_2$ contains the terms with derivatives of $\chi_2$. We can estimate $h^3\partial_x^3 q\partial_\xi^3 a$ by estimating

$$e_2(x,\xi) = -h^3\left(\left(\frac{\tilde{h}}{h}\right)^2 h^2V'''(x)\Lambda'''\Big(\big(\frac{\tilde{h}}{h}\big)^{2/3}\xi\Big)\Lambda\Big(\big(\frac{\tilde{h}}{h}\big)^{1/3}x\Big)\right). \tag{4.4.40}$$

The goal for this section is the following lemma,

**Lemma 4.4.41.** *For $e_2$ as defined in* (4.4.40) *we have*

$$|(e_2\circ\mathcal{B})(X,\Xi)| \le C_{e_2}\tilde{h}^{4/3}h^{4/3}g_1(X,\Xi) \tag{4.4.42}$$

*for a positive constant $C_{e_2} > 0$ independent of $h$ and $\tilde{h}$.*

*Proof.* Recall from the construction of $\Lambda$ that for $|s| < \varepsilon_\lambda$, $\Lambda'''(s) = 0$. This means we just need show 4.4.42 when $|\Xi| \ge \varepsilon_\lambda$. Recall from Lemma 4.4.26 that

$$-h^2V'(x)\Lambda(x) \ge \frac{2}{5}\left(\frac{E_1}{E_3}x\Lambda(X) + x^3\Lambda(X)\right). \tag{4.4.43}$$

Our goal will be to show that

$$\left|h^3\left(\left(\frac{\tilde{h}}{h}\right)^2 h^2V'''(x)\Lambda'''\Big(\big(\frac{\tilde{h}}{h}\big)^{2/3}\xi\Big)\Lambda\Big(\big(\frac{\tilde{h}}{h}\big)^{1/3}x\Big)\right)\right| \le \tilde{h}^{4/3}g \tag{4.4.44}$$

where

$$g = \left(2h^{2/3}\tilde{h}^{1/3}\xi\Lambda(\Xi)\lambda(X) - h^{1/3}\tilde{h}^{2/3}h^2V'(x)\Lambda(X)\lambda(\Xi)\right).$$

Noting that $\Lambda'''(\Xi) = 0$ for $|\Xi| \le \varepsilon_\lambda$ and using Lemma 4.4.26, (4.4.44) holds if

$$\left|h\tilde{h}^2(h^2V'''(x))\Lambda'''(\Xi)\Lambda(X)\right| \le K\tilde{h}^{4/3}\left(h^{4/3}\tilde{h}^{-1/3} + h^{1/3}\tilde{h}^{2/3}\frac{2}{5}\left(\frac{E_1}{E_3}x\Lambda(X) + x^3\Lambda(X)\right)\Lambda'(\Xi)\right).$$

From the calculations on $h^2V'(x)$ we have

$$-h^2V'''(x) = 24(x - E_2/3E_3). \tag{4.4.45}$$

Noting that $|\Lambda'''(X)| \leq C''_{\varepsilon_\lambda}\Lambda'(X)$ implies that (4.4.42) holds if

$$\left|24h\tilde{h}^2\left(x - \frac{E_2}{3E_3}\right)\Lambda(X)\right| \leq C_{e_2}\tilde{h}^{4/3}\left(h^{4/3}\tilde{h}^{-1/3} + h^{1/3}\tilde{h}^{2/3}\frac{2}{5}\left(\frac{E_1}{E_3}x\Lambda(X) + x^3\Lambda(X)\right)\right). \tag{4.4.46}$$

After the blowdown map, (4.4.46) holds if

$$\begin{aligned}24h^{4/3}\tilde{h}^{5/3}X\Lambda(X) &+ \left|24h\tilde{h}^2\frac{E_2}{3E_3}\Lambda(X)\right| \\ &\leq C_{e_2}\tilde{h}^{4/3}\left(h^{4/3}\tilde{h}^{-1/3} + h^{2/3}\tilde{h}^{1/3}\frac{2}{5}\frac{E_1}{E_3}X\Lambda(X) + h^{4/3}\tilde{h}^{-1/3}\frac{2}{5}X^3\Lambda(X)\right).\end{aligned} \tag{4.4.47}$$

We will first deal with the $24h^{4/3}\tilde{h}^{5/3}X\Lambda(X)$ term. If $|X| \leq \varepsilon_\lambda$, then

$$24h^{4/3}\tilde{h}^{5/3}X\Lambda(X) \leq 24h^{4/3}\tilde{h}^{5/3}\varepsilon_\lambda^2. \tag{4.4.48}$$

If $|X| \geq \varepsilon_\lambda$, then

$$24h^{4/3}\tilde{h}^{5/3}X\Lambda(X) \leq \frac{24}{\varepsilon_\lambda^2}h^{4/3}\tilde{h}^{5/3}X^3\Lambda(X). \tag{4.4.49}$$

Combining (4.4.48) and (4.4.49) gives

$$24h^{4/3}\tilde{h}^{5/3}X\Lambda(X) \leq C'_{e_2}\tilde{h}^2(h^{4/3}\tilde{h}^{-1/3} + h^{4/3}\tilde{h}^{-1/3}\frac{2}{5}X^3\Lambda(X)) \tag{4.4.50}$$

for a positive constant $C'_{e_2}$ independent of $h$.

Now we will handle the $\left|24h\tilde{h}^2\frac{E_2}{3E_3}\Lambda(X)\right|$ term. First note that $|\Lambda(X)| \leq |X|$. If $|\frac{8E_2}{3E_3}X| \leq h^{1/3}$, then

$$\left|24h\tilde{h}^2\frac{E_2}{3E_3}\Lambda(X)\right| \leq 24h^{4/3}\tilde{h}^2. \tag{4.4.51}$$

If $|\frac{8E_2}{3E_3}X| \geq h^{1/3}$, then $|X| \geq h^{1/3}\frac{3E_3}{8E_2}$. Noting that $X\Lambda(X) \geq 0$, gives

$$\left|24h\tilde{h}^2\frac{E_2}{3E_3}\Lambda(X)\right| \leq h^{2/3}\tilde{h}^2\left(\frac{8E_2}{3E_3}\right)^2 X\Lambda(X).$$

Notice that

$$h^{2/3}\tilde{h}^2\left(\frac{8E_2}{3E_3}\right)^2 X\Lambda(X) \leq \tilde{h}^{4/3}h^{2/3}\tilde{h}^{1/3}\frac{2E_1}{5E_2}X\Lambda(X),$$

if

$$\tilde{h}^{2-5/3}\left(\frac{8E_2}{3E_3}\right)^2 \leq \frac{2E_1}{5E_3}.$$

This is equivalent to

$$\frac{2}{5}E_1 - \tilde{h}^{1/3}\frac{64E_2^3}{9E_3} \geq 0.$$

Multiplying by 5/2 gives that we need

$$E_1 - \tilde{h}^{1/3}\frac{160E_2^2}{9E_3} \geq 0.$$

Recall from the proof of Lemma 4.4.26 that

$$(1-k)E_1 - \frac{E_2^2}{4(1-\tilde{k})E_3} \geq 0$$

for $k = \tilde{k} = \frac{1}{10}$. Taking $\tilde{h}$ sufficiently small gives

$$h^{2/3}\tilde{h}^2\left(\frac{8E_2}{3E_3}\right)^2 X\Lambda(X) \leq \tilde{h}^{4/3}h^{2/3}\tilde{h}^{1/3}\frac{2E_1}{5E_3}X\Lambda(X). \tag{4.4.52}$$

Combining the estimates in (4.4.51) and (4.4.52) gives

$$\left|24h\tilde{h}^2\frac{E_2}{3E_3}\Lambda(X)\right| \leq C''_{e_2}\tilde{h}^{4/3}\left(h^{4/3}\tilde{h}^{-1/3} + h^{2/3}\tilde{h}^{1/3}\frac{2}{5}\frac{E_1}{E_3}X\Lambda(X)\right) \tag{4.4.53}$$

for a positive constant $C''_{e_2}$ independent of $h$.

Now combining (4.4.50) and (4.4.53) gives

$$\begin{aligned}|e_2 \circ \mathcal{B}\,| &\leq 24h^{4/3}\tilde{h}^{5/3}X\Lambda(X) + \left|24h\tilde{h}^2\frac{E_2}{3E_3}\Lambda(X)\right| \\ &\leq (C'_{e_2} + C''_{e_2})\tilde{h}^{4/3}\left(h^{4/3}\tilde{h}^{-1/3} + h^{2/3}\tilde{h}^{1/3}\frac{2}{5}\frac{E_1}{E_3}X\Lambda(X) + h^{4/3}\tilde{h}^{-1/3}\frac{2}{5}X^3\Lambda(X)\right) \\ &\leq C_{e_2}\tilde{h}^{4/3}h^{4/3}g_1\end{aligned}$$

where $C_{e_2} = C'_{e_2} + C''_{e_2} > 0$ and is independent of $h$. □

Lemma 4.4.41 implies that

$$\mathrm{Op}\,^w_{\tilde{h}}(g \circ \mathcal{B}\,) + \mathrm{Op}\,^w_{\tilde{h}}(e_2 \circ \mathcal{B}\,) = h^{4/3}\tilde{h}^{-1/3}\mathrm{Op}\,^w_{\tilde{h}}(g_1(1 + \tilde{r})) \tag{4.4.54}$$

where $\tilde{r} = \mathcal{O}_{\mathcal{S}_{1/3,2/3}}(\tilde{h}^{5/3})$.

### 4.4.6 Final estimate

We will use (4.4.54) and Lemma 4.4.39 to prove the following lemma,

**Lemma 4.4.55.** *For $\delta > 0$ sufficiently small and $h > 0$ sufficiently small, let $\varphi \in \mathcal{S}(\mathbb{R})$ have compact support in $\{|(x,\xi)| \leq \delta\}$. Then there exists a $C > 0$ such that*

$$||(Q_1 - z)\varphi^w u|| \geq Ch^{4/3}||\varphi^w u||,\ z \in [\tilde{M} - \delta, \tilde{M} + \delta]. \tag{4.4.56}$$

*Proof.* Let $v = \varphi^w u$ for $\varphi$ chosen with support inside the set where $\chi_2(x)\chi_2(\xi) = 1$. Thus $r$ and $r_2$ are supported away from the support of $\varphi$. Recall that

$$\langle \mathrm{Op}\,^w_h(s)u, u\rangle = \langle \mathrm{Op}\,^w_{\tilde{h}}(s \circ \mathcal{B}\,)(X,\Xi)T_{h,\tilde{h}}u(X), T_{h,\tilde{h}}u(X)\rangle = \langle \mathrm{Op}\,^w_{\tilde{h}}(s \circ \mathcal{B}\,)(X,\Xi)u(X), u(X)\rangle$$

for $s \in \mathcal{S}_{1/3,2/3}$. Hence,

$$\begin{aligned} i\langle[Q_1 - z, a^w]v, v\rangle &= h\langle \operatorname{Op}_h^w(\{q, a\})v, v\rangle + \langle \operatorname{Op}_h^w(e_2)v, v\rangle + \mathcal{O}(h^\infty)||v||_{L^2}^2 \\ &= \langle \operatorname{Op}_h^w(g)v, v\rangle + \langle \operatorname{Op}_h^w(e_2)v, v\rangle + \mathcal{O}(h^\infty)||v||_{L^2}^2 \\ &= \langle \operatorname{Op}_{\tilde{h}}^w(g \circ \mathcal{B})v, v\rangle + \langle \operatorname{Op}_{\tilde{h}}^w(e_2 \circ \mathcal{B})v, v\rangle + \mathcal{O}(h^\infty)||v||_{L^2}^2 \\ &= h^{4/3}\tilde{h}^{-1/3}\langle \operatorname{Op}_{\tilde{h}}^w(g_1(1+\tilde{r}))v, v\rangle + \mathcal{O}(h^\infty)||v||_{L^2}^2, \text{ by (4.4.54)} \\ &\geq Ch^{4/3}\tilde{h}||v||_{L^2}^2, \text{ by Lemma 4.4.39} \end{aligned}$$

for $\tilde{h}$ sufficiently small. We are almost finished. Notice that

$$|a(x,\xi;h)| = \left|\Lambda\left(\left(\frac{\tilde{h}}{h}\right)^{1/3} x\right)\Lambda\left(\left(\frac{\tilde{h}}{h}\right)^{2/3}\xi\right)\chi_2(x)\chi_2(\xi)\right| \leq C$$

Hence,

$$|\langle[Q_1 - z, a^w]v, v\rangle| \leq \|(Q_1 - z)v\|_{L^2}\,\|a^w v\|_{L^2} \leq C\|(Q_1 - z)v\|_{L^2}\,\|v\|_{L^2}.$$

So for fixed $\tilde{h} > 0$ sufficiently small,

$$\|(Q_1 - z)v\|_{L^2} \geq Ch^{4/3}\|v\|_{L^2}.$$

□

# APPENDIX

## A.1 Fourier Transform

In this section we will define the Fourier transform and semiclassical Fourier transform and explain the useful properties.

**Definition A.1.1.** *If $\varphi \in \mathscr{S}(\mathbb{R}^n)$, then the Fourier transform of $\varphi$ is*

$$\mathscr{F}\varphi(\xi) = \hat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\varphi(x)dx.$$

*The inverse Fourier transform of a function $\varphi \in \mathscr{S}(\mathbb{R}^n)$ is given by*

$$\mathscr{F}^{-1}\varphi(x) = \check{\varphi}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\varphi(\xi)dx.$$

**Definition A.1.2.** *If $\varphi \in \mathscr{S}(\mathbb{R}^n)$, then the semiclassical Fourier transform of $\varphi$ is*

$$\mathscr{F}_h\varphi(\xi) = \hat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x,\xi\rangle}\varphi(x)dx.$$

*The inverse semiclassical Fourier transform of a function $\varphi \in \mathscr{S}(\mathbb{R}^n)$ is given by*

$$\mathscr{F}_h^{-1}\varphi(x) = \check{\varphi}(x) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle}\varphi(\xi)dx.$$

The Fourier Transform has the following nice properties,

**Theorem A.1.3.** *For $\varphi \in \mathscr{S}(\mathbb{R}^n)$*

*(i)*

$$\mathscr{F}_h^{-1}\mathscr{F}_h\varphi(x) = \varphi(x)$$

*(ii)*

$$(hD_\xi)^\alpha(\mathscr{F}_h\varphi) = \mathscr{F}_h((-x)^\alpha\varphi)$$

*(iii)*

$$\mathscr{F}_h((hD_x)^\alpha \varphi) = \xi^\alpha \mathscr{F}_h \varphi$$

*(iv)*

$$||\varphi||_{L^2}^2 = \frac{1}{(2\pi h)^n} ||\mathscr{F}_h \varphi||_{L^2}.$$

A useful property of this is that,

$$(hD_x)^\alpha \varphi = \mathscr{F}_h^{-1}(\xi^\alpha \mathscr{F}_h \varphi)(x). \tag{A.1.4}$$

This allows the definition of Sobolev Spaces with non-integer values. Sobolev norms are defined in the following way,

**Definition A.1.5** ($L^2$-based Sobolev Norm)**.** *Let $k \in \mathbb{N}$. Then the Sobolev norm of a function $u$ on $\mathbb{R}^n$ is given by*

$$||u||_{H^k} := \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |D^\alpha u|^2 dx \right)^{1/2}.$$

However, using the idea behind (A.1.4) and psuedo-differential operators and property (*iv*) we can extend the idea of Sobolev norms to non-integer $k$.

**Definition A.1.6** (Generalized Sobolev Norm)**.** *Let $s \in \mathbb{R}_+$. Then the Sobolev norm of a function $u$ on $\mathbb{R}^n$ is given by*

$$||u||_{H^s} := \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\mathscr{F} u(\xi)|^2 d\xi \right)^{1/2}$$

*where $\langle \xi \rangle^{2s} = (1 + \xi^2)^s$.*

**Remark A.1.7.** *Notice that there is a factor of $\langle \xi \rangle^{2s} \to \infty$ as $|\xi| \to \infty$, so high frequency estimates ($|\xi|$ large) are what determines the regularity.*

**Theorem A.1.8** (Uncertainty Principle)**.** *We have*

$$\frac{h}{2}||f||_{L^2}||\mathscr{F}_h f||_{L^2} \leq ||x_j f||_{L^2}||\xi_j \mathscr{F}_h f||_{L^2}.$$

The uncertainty principle implies that you cannot arbitrarily localize in phase space. This is an important barrier to the local smoothing estimates of the Schrödinger Equation.

## A.2 Stationary Phase

**Theorem A.2.1** (Rapid Decay Lemma 3.10 in [Zwo12])**.** *Given functions* $a \in \mathcal{C}_c^\infty(\mathbb{R}), \varphi \in \mathcal{C}^\infty(\mathbb{R})$ *and* $h > 0$ *let*

$$I_h = I_h(a, \varphi) := \int_{\mathbb{R}} e^{\frac{i\varphi}{h}} a dx.$$

*If* $\varphi'(x) \neq 0$ *on* $\operatorname{supp}(a)$, *then*

$$I_h = \mathcal{O}(h^\infty) \text{ as } h \to 0.$$

*Proof.* The proof is based on integrating by parts multiple times, so we will follow the proof from [Zwo12] almost line for line. To prove the theorem we will need to show that for each positive integer $N$, there is a $C_N$ such that $|I_h| \leq C_N h^N$ for all $0 < h \leq 1$. Let

$$L := \frac{h}{i}\frac{1}{\varphi'}\partial_x.$$

Note that this is where we require that $\varphi'(x) \neq 0$, so that $L$ is defined for all $x \in \operatorname{supp}(a)$. $L$ was constructed so that

$$Le^{\frac{i\varphi}{h}} = e^{\frac{i\varphi}{h}}.$$

This implies that applying $L$ to $e^{\frac{i\varphi}{h}}$ returns $e^{\frac{i\varphi}{h}}$ so,

$$|I_h| = \left| \int_{\mathbb{R}} (L^N e^{\frac{i\varphi}{h}}) a dx \right| = \left| \int_{\mathbb{R}} e^{\frac{i\varphi}{h}} (L^*)^N a dx \right|$$

where $L^*$ is the adjoint of $L$. Since $a$ is smooth compactly supported and $\varphi$ is smooth,

$$L^* a = -\frac{h}{i}\partial_x(\frac{a}{\varphi'})$$

is of size $h$. Hence, $|I_h| \leq C_N h^N$. □

This theorem tells us that if the phase is not stationary, then we should expect rapid decay as $h \to 0$. This is because the integral will oscillate rapidly producing cancellations in the integral. If $\varphi' = 0$ at a single point, then we should expect the integral to be determined by the value at this point, since outside of this point the integral will be oscillating rapidly and produce cancellations.

**Theorem A.2.2.** *Let $a \in \mathcal{C}_c^\infty(\mathbb{R})$. Suppose that $x_0 \in \operatorname{supp}(a)$ and $\varphi'(x_0) = 0, \varphi''(x_0) \neq 0$. Additionally, suppose $\varphi'(x) \neq 0$ for all $x \in \operatorname{supp}(a) \setminus \{x_0\}$. Then,*

$$\int_{\mathbb{R}} e^{\frac{i\varphi}{h}} a dx = (2\pi h)^{1/2} |\varphi''(x_0)|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}(\varphi''(x_0))} e^{\frac{i\varphi(x_0)}{h}} a(x_0) + O(h^{3/2}).$$

### A.3 Laplacian

For a given Riemannian Manifold $X$ without boundary with metric $g$ and local coordinates $x_1, \cdots, x_n$ the Laplace-Beltrami operator is given by

$$\Delta_g = \sum_{i,j} \frac{1}{\sqrt{|g|}} \partial_i g^{ij} \sqrt{|g|} \partial_j$$

where $g^{ij}$ is the $i,j$-th entry of the inverse of the metric $g$ and $|g|$ is the determinate of the metric tensor. The volume form is given by $dVol = \sqrt{|g|} dx_1 \cdots dx_n$. Note that $\Delta_g$ is essentially self adjoint since for $u, v \in \mathscr{S}(X)$,

$$\begin{aligned}
\int_X \Delta_g u\bar{v}\, dVol &= \int_X \sum_{i,j} \big(\frac{1}{\sqrt{|g|}}\partial_i g^{ij}\sqrt{|g|}\partial_j u\big)\bar{v}\sqrt{|g|}dx_1\cdots dx_n \\
&= \int_X \sum_{i,j} \big(\partial_i g^{ij}\sqrt{|g|}\partial_j u\big)\bar{v}dx_1\cdots dx_n \\
&= -\int_X \sum_{i,j} \big(g^{ij}\sqrt{|g|}\partial_j u\big)\partial_i\bar{v}dx_1\cdots dx_n \\
&= -\int_X \sum_{i,j} \partial_j u\big(\sqrt{|g|}g^{ij}\partial_i\bar{v}\big)dx_1\cdots dx_n \\
&= \int_X \sum_{i,j} u\big(\partial_j\sqrt{|g|}g^{ij}\partial_i\bar{v}\big)dx_1\cdots dx_n \\
&= \int_X u\sum_{i,j} \frac{1}{\sqrt{|g|}}\big(\partial_j\sqrt{|g|}g^{ij}\partial_i\bar{v}\big)\sqrt{|g|}dx_1\cdots dx_n \\
&= \int_X u\overline{\Delta_g v}\, dVol
\end{aligned}$$

since $g$ is symmetric. In the multi-warped product case in this thesis the metric is

$$g = dx^2 + A_-(x)^2 d{\theta_+}^2 + A_+(x)^2 d{\theta_-}^2.$$

Then,

$$\Delta_g = \partial_x^2 + \frac{A_-' + A_+'}{A_- A_+}\partial_x + A_+(x)^{-2}\partial_{\theta_+}^2 + A_-(x)^{-2}\partial_{\theta_-}^2$$

and

$$dVol = A_+(x)A_-(x)dx d\theta_+ d\theta_-.$$

There are two issues with this Laplacian. There is a first order derivative term and when we integrate we need to be careful to include the volume form. We solve both of these issues by conjugating by $T = (A_1(x)A_2(x))^{1/2}$ and studying the operator $\tilde{\Delta} = T\Delta T^{-1}$ instead of $\Delta$.

After the conjugation

$$\tilde{\Delta} = \partial_x^2 + A_-(x)^{-2}\partial_{\theta_+}^2 + A_+(x)^{-2}\partial_{\theta_-}^2 + V_1(x)$$

for a potential term $V_1(x)$. This eliminates the first order derivative at the cost of an added potential term $V_1(x)$. However, this term $V_1(x)$ in our case does not harm the local smoothing estimates. The removal of the first order derivative and volume form make studying $\tilde{\Delta}$ easier.

We will go over why we can study $\tilde{\Delta}$ instead of $\Delta$. Note that $T : L^2(X, dVol) \to L^2(X, dxd\theta_- d\theta_+)$ is an isometry since

$$\int_X |u|^2 dVol = \int_X |u|^2 A_+(x)A_-(x)dxd\theta_- d\theta_+ = \int_X |Tu|^2 dxd\theta_- d\theta_+.$$

However, $T$ is not an isometry on $H^s$ norms when $s \neq 0$. We do not have $||Tu||_{H^s(X,dxd\theta_-d\theta_+)} = ||u||_{H^s(X,dVol)}$, since $x$-derivatives do hit the $(A_+(x)A_-(x))^{1/2}$ terms. However, since $g$ is asymptotically Euclidean, there are some positive constants $C_T$ and $C_{T^{-1}}$ such that $||Tu||_{H^s(X,dxd\theta_-d\theta_+)} \leq C_T||u||_{H^s(X,dVol)}$ and $||\tilde{u}||_{H^s(X,dxd\theta_-d\theta_+)} \leq C_{T^{-1}}||T^{-1}\tilde{u}||_{H^s(X,dVol)}$ for $u \in H^s(X, dVol)$ and $\tilde{u} \in H^s(X, dxd\theta_- d\theta_+)$. Now, suppose $\tilde{u}$ solves

$$\begin{cases} (D_t - \tilde{\Delta})\tilde{u}(t,x) = 0 \\ \tilde{u}(0,x) = \tilde{u}_0(x) \end{cases} \tag{A.3.1}$$

for $\tilde{u}_0(x) \in \mathscr{S}(X, dxd\theta_- d\theta_+)$. If $u = T^{-1}\tilde{u}$, then

$$0 = (D_t - \tilde{\Delta})\tilde{u}(t,x) = (TD_tT^{-1} - T\Delta T^{-1})\tilde{u} = T(D_t - \Delta)T^{-1}Tu = T(D_t - \Delta)u.$$

and $u_0(x) = T^{-1}\tilde{u}_0(x) \in H^{1/2}(X, dVol)$. Therefore $u$ is a solution to

$$\begin{cases} (D_t - \Delta)u(t,x) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{A.3.2}$$

This implies that if we can show

$$\int_0^T ||\langle x\rangle^{-3/2}\tilde{u}||^2_{H^1(X,dxd\theta_-d\theta_+)}dt \leq C||\tilde{u}_0||^2_{H^{2/3}(X,dxd\theta_-d\theta_+)}$$

for some constant $C > 0$ then,

$$\int_0^T ||\langle x\rangle^{-3/2}u||^2_{H^1(X,dVol)}dt \leq C'||u_0||^2_{H^{2/3}(X,dVol)}$$

for some constants $C' > 0$, which is the estimate in Theorem 4.2.1. Hence, we can get the desired local smoothing estimates by studying solutions to (A.3.1).

## A.4 Local Smoothing on $\mathbb{R}$ from Propagation

**Theorem A.4.1.** *Suppose $u_0(x) \in \mathcal{C}_c^\infty(\mathbb{R})$ and $u(t,x)$ solves*

$$\begin{cases} (D_t + D_x^m)u(t,x) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

*where $D_t = \frac{1}{i}\partial_t$ and $D_x^m = \frac{1}{i^m}\partial_x^m$. Suppose $I$ is a compact interval. Then, for every $T > 0$ there exists a constant $C_T$ such that*

$$\int_0^T \int_I |\langle D_x\rangle^{(m-1)/2}u(t,x)|^2 dxdt \leq C_T||u_0||^2_{L^2}.$$

*Proof.* Suppose $u_0(x) \in \mathcal{C}_c^\infty(\mathbb{R})$ and $u(t,x)$ solves

$$\begin{cases} (D_t + D_x^m)u(t,x) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

where $D_t = \frac{1}{i}\partial_t$ and $D_x^m = \frac{1}{i^m}\partial_x^m$. Let $I = [I_a, I_b]$ where $I_b > I_a$ and $\text{supp}\,(u_0) \subseteq [U_a, U_b]$ where $U_b > U_a$. Let $\chi^+ \in \mathcal{C}_c^\infty([1/2, 2])$ ,$\chi^- \in \mathcal{C}_c^\infty([-2, -1/2])$ and $\chi \in \mathcal{C}_c^\infty([-1, 1])$ such that

$0 \le \chi^+, \chi^-, \chi \le 1$ and

$$1 = \sum_{j=0}^{\infty} \chi^+(\xi/2^j) + \sum_{j=0}^{\infty} \chi^-(\xi/2^j) + \chi.$$

Define $\chi_j^+(\xi) = \chi^+(\xi/2^j)$ and $\chi_j^-(\xi) = \chi^-(\xi/2^j)$. Let $u_j^+ = \mathscr{F}^{-1}(\chi_j^+\hat{u})$, $u_j^- = \mathscr{F}^{-1}(\chi_j^-\hat{u})$, $u_{j,0}^+ = \mathscr{F}^{-1}(\chi_j^+\hat{u}_0)$, $u_{j,0}^- = \mathscr{F}^{-1}(\chi_j^+\hat{u}_0)$, $\tilde{u} = \mathscr{F}^{-1}(\chi\hat{u})$ and $\tilde{u}_0 = \mathscr{F}^{-1}(\chi\hat{u}_0)$ where we take the Fourier transform in $x$ only. Then,

$$u(t,x) = \sum_{j=0}^{\infty} u_j^+ + \sum_{j=0}^{\infty} u_j^- + \tilde{u}$$

and

$$u_0(x) = \sum_{j=0}^{\infty} u_{j,0}^+ + \sum_{j=0}^{\infty} u_{j,0}^- + \tilde{u}_0.$$

This division ensures that $\hat{u}_j^+$ has support on $[1/2(2^j), 2(2^j)]$. We will focus on $u_j^+$ and the case of $u_j^-$ will follow similarly. Note that $u_j^+$ solves $(D_t + D_x^m)u_j^+(t,x) = 0$ with initial condition $u_j^+(0,x) = u_{j,0}^+(x)$. Let

$$A_j = \int_0^T \int_I |\langle D_x\rangle^{(m-1)/2} u_j^+(t,x)|^2 dxdt.$$

Then,

$$\begin{aligned}
A_j &= \int_0^T \int_I |\langle D_x\rangle^{(m-1)/2} u_j^+(t,x)|^2 dxdt \\
&= \frac{1}{2\pi} \int_0^T \int_I \left| \int_{\mathbb{R}} e^{ix\xi} \langle\xi\rangle^{(m-1)/2} \hat{u}_j^+(t,\xi) d\xi \right|^2 dxdt \\
&= \frac{1}{2\pi} \int_0^T \int_I \left| \int_{\mathbb{R}} e^{ix\xi} (1+\xi^2)^{(m-1)/4} e^{-it\xi^m} \hat{u}_{j,0}^+(\xi) d\xi \right|^2 dxdt.
\end{aligned}$$

Now $\hat{u}_{j,0}^+$ is supported near $\xi = 2^j$. Specifically, $\operatorname{supp}(\hat{u}_{j,0}^+) \subseteq [\frac{1}{2}2^j, 2(2^j)]$. Let $h = 1/2^j$, so that $\hat{u}_0^+$ is supported near $h^{-1}$ and $\operatorname{supp}(\hat{u}_0^+) \subseteq [\frac{1}{2}h^{-1}, 2h^{-1}]$. Then,

$$\begin{aligned} A_j &= \frac{1}{2\pi}\int_0^T \int_I \left| \int_{\mathbb{R}} e^{ix\xi}(1+\xi^2)^{(m-1)/4} e^{-it\xi^m} \hat{u}_{j,0}^+(\xi) d\xi \right|^2 dxdt \\ &\leq \frac{1}{2\pi}(1+4(h^{-1})^2)^{(m-1)/2} \int_0^T \int_I \left| \int_{\mathbb{R}} e^{ix\xi} e^{-it\xi^m} \hat{u}_{j,0}^+(\xi) d\xi \right|^2 dxdt \\ &= \frac{1}{2\pi}(1+4(h^{-1})^2)^{(m-1)/2} \int_0^T \int_I \left| \int_{\mathbb{R}}\int_{\mathbb{R}} e^{ix\xi} e^{-it\xi^m} e^{-iy\xi} u_{j,0}^+(y) dy d\xi \right|^2 dxdt \\ &= \frac{1}{2\pi}(1+4(h^{-1})^2)^{(m-1)/2} \int_0^T \int_I \left| \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i[(x-y)\xi - t\xi^m]} u_{j,0}^+(y) dy d\xi \right|^2 dxdt. \end{aligned}$$

Take $\chi_0 \in \mathcal{C}_c^\infty$ such that $\chi_0 = 1$ on support of $\chi^+$, $0 \leq \chi_0 \leq 1$, and $\operatorname{supp}(\chi_0) = [1/3, 3]$. Let $\chi_{j,0}(\xi) = \chi_0(\xi/2^j) = \chi_0(h\xi)$. Then,

$$\begin{aligned} A_j &\leq \frac{1}{2\pi}(1+4(h^{-1})^2)^{(m-1)/2} \int_0^T \int_I \left| \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i[(x-y)\xi - t\xi^m]} u_{j,0}^+(y) dy d\xi \right|^2 dxdt \\ &= \frac{1}{2\pi}(1+4(h^{-1})^2)^{(m-1)/2} \int_0^T \int_I \left| \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i[(x-y)\xi - t\xi^m]} \chi_{j,0}(\xi) u_{j,0}^+(y) dy d\xi \right|^2 dxdt \\ &= \frac{1}{2\pi}(1+4(h^{-1})^2)^{(m-1)/2} \int_0^T \int_I \left| \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i[(x-y)\xi - t\xi^m]} \chi_{j,0}(\xi) d\xi u_{j,0}^+(y) dy \right|^2 dxdt. \end{aligned}$$

The introduction of $\chi_0$ is so that we can examine the integral kernel

$$B_1 = \int_{\mathbb{R}} e^{i[(x-y)\xi - t\xi^m]} \chi_{j,0}(\xi) d\xi.$$

Set $\xi = \eta/h$. Then,

$$\begin{aligned} B_1 &= \frac{1}{h}\int_{\mathbb{R}} e^{\frac{i}{h}[(x-y)\eta - t\eta^m/h^{m-1}]} \chi_{j,0}(\eta/h) d\eta \\ &= \frac{1}{h}\int_{\mathbb{R}} e^{\frac{i}{h}\varphi} \chi_0(\eta) d\eta \end{aligned}$$

for $\varphi = [(x-y)\eta - t\eta^m/h^{m-1}]$. Now, $\varphi_\eta = 0$ when $\eta = h(\frac{(x-y)}{mt})^{m-1}$. Take a smooth cutoff function $\tilde{\chi}_0$ such that $\tilde{\chi}_0 = 1$ on supp $(\chi_0)$, $0 \leq \tilde{\chi}_0 \leq 1$ and supp $(\tilde{\chi}_0) = [1/4, 4]$.

$$\begin{aligned} B_1 &= \frac{1}{h}\tilde{\chi}_0\left(h\left(\frac{(x-y)}{mt}\right)^{m-1}\right)\int_{\mathbb{R}} e^{\frac{i}{h}\varphi}\chi_{j,0}(\eta/h)d\eta \\ &\quad + \frac{1}{h}\left(1-\tilde{\chi}_0\left(h\left(\frac{(x-y)}{mt}\right)^{m-1}\right)\right)\int_{\mathbb{R}} e^{\frac{i}{h}\varphi}\chi_{j,0}(\eta/h)d\eta \\ &= \frac{1}{h}\tilde{\chi}_0\left(h\left(\frac{(x-y)}{mt}\right)^{m-1}\right)\int_{\mathbb{R}} e^{\frac{i}{h}\varphi}\chi_0(\eta)d\eta + \mathcal{O}(h^\infty). \end{aligned}$$

This follows from rapid decay as $h \to 0$ since $\varphi_\eta \neq 0$ on the support of

$$\left(1 - \tilde{\chi}_0\left(h\left(\frac{(x-y)}{mt}\right)^{m-1}\right)\right)\chi_0(\eta).$$

Using the calculations for $B_1$ gives

$$\begin{aligned} A_j \leq& \frac{1}{2\pi}(1+4(h^{-1})^2)^{(m-1)/2}\int_0^T\int_I\left|\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i[(x-y)\xi - t\xi^m]}\chi_{j,0}(\xi)d\xi u^+_{j,0}(y)dy\right|^2 dxdt \\ \leq& \frac{1}{2\pi}(1+4h^{-2})^{(m-1)/2}\int_0^T\int_I\left|\int_{\mathbb{R}}\frac{1}{h}\tilde{\chi}_0\left(h\left(\frac{(x-y)}{mt}\right)^{m-1}\right)u^+_{j,0}(y)dy\right|^2 dxdt + \mathcal{O}(h^\infty)||u^+_{j,0}||^2_{L^2} \\ =& \frac{1}{2\pi}(1+4h^{-2})^{(m-1)/2}\int_0^T\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\frac{1}{h}\chi_I(x)\tilde{\chi}_0\left(h\left(\frac{(x-y)}{mt}\right)^{m-1}\right)u^+_{j,0}(y)dy\right|^2 dxdt + \mathcal{O}(h^\infty)||u^+_{j,0}||^2_{L^2} \end{aligned}$$

where $\chi_I(x) = 1$ for $x \in I$ and $\chi_I = 0$ for $x \notin I$.

The issue here is that $u_{j,0}$ does not have compact support. Let $\Phi \in \mathcal{C}_c^\infty$ such that $\Phi(x) = 1$ on $[U_a, U_b]$ and supp $(\Phi) = [U_a - \epsilon, U_b + \epsilon]$ for $\epsilon > 0$ small and $\tilde{\Phi} \in \mathcal{C}_c^\infty$ such that $\tilde{\Phi} = 1$ on supp $(\Phi)$,supp $(\tilde{\Phi}) = [U_a - 2\epsilon, U_b + 2\epsilon]$ and $0 \leq \Phi, \tilde{\Phi} \leq 1$. Let

$$B_2 = \int_{\mathbb{R}}\frac{1}{h}\chi_I(x)\tilde{\chi}_0\left(h\left(\frac{(x-y)}{mt}\right)^{m-1}\right)(1-\tilde{\Phi}(y))u_{j,0}(y)dy.$$

Then,

$$B_2 = \int_{\mathbb{R}} \frac{1}{h}\chi_I(x)\tilde{\chi}_0\Bigg(h\Big(\frac{(x-y)}{mt}\Big)^{m-1}\Bigg)(1-\tilde{\Phi}(y))\int_{\mathbb{R}}\int_{\mathbb{R}} e^{iy\xi}\chi^+(h\xi)e^{-iz\xi}u_0(z)dzd\xi dy.$$

Let $\xi = \eta/h$. Then,

$$\begin{aligned}
B_2 &= \int_{\mathbb{R}} \frac{1}{h}\chi_I(x)\tilde{\chi}_0\Bigg(h\Big(\frac{(x-y)}{mt}\Big)^{m-1}\Bigg)(1-\tilde{\Phi}(y))\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{h}e^{i(y-z)\eta/h}\chi^+(\eta)\Phi(z)u_0(z)dzd\eta dy \\
&= \int_{\mathbb{R}} \frac{1}{h}\chi_I(x)\tilde{\chi}_0\Bigg(h\Big(\frac{(x-y)}{mt}\Big)^{m-1}\Bigg)\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} \frac{1}{h}e^{i(y-z)\eta/h}(1-\tilde{\Phi}(y))\Phi(z)\chi^+(\eta)d\eta\Big)u_0(z)dzdy \\
&= \mathcal{O}(h^\infty)||u_0||^2_{L^2}
\end{aligned}$$

due to rapid decay as $h \to 0$, since $(1-\tilde{\Phi}(y))\Phi(z) = 0$ when $y = z$. Hence,

$$A_j \le \frac{1}{2\pi}(1+4h^{-2})^{(m-1)/2}\int_0^T\int_{\mathbb{R}}\Bigg|\int_{\mathbb{R}} \frac{1}{h}\chi_I(x)\tilde{\chi}_0\Bigg(h\Big(\frac{(x-y)}{mt}\Big)^{m-1}\Bigg)\tilde{\Phi}(y)u_{j,0}(y)dy\Bigg|^2 dxdt$$
$$+ \mathcal{O}(h^\infty)||u^+_{j,0}||^2_{L^2} + \mathcal{O}(h^\infty)||u_0||^2_{L^2}.$$

For $\chi_I(x)\tilde{\chi}_0\Big(h\big(\frac{(x-y)}{mt}\big)^{m-1}\Big)\tilde{\Phi}(y)u_{j,0}(y) \neq 0$ we need $h(\frac{(x-y)}{mt})^{1/(m-1)} \in [1/4, 4]$. Hence, $(x-y) \in [(h^{-1}/4)^{m-1}t, (4h^{-1})^{m-1}t]$. Since $\tilde{\Phi} \neq 0$ for $y \in [U_a - 2\varepsilon, U_b + 2\varepsilon]$ we need $x \in [(h^{-1}/4)^{m-1}t + (U_a - 2\epsilon), (4h^{-1})^{m-1}t + (U_b + 2\epsilon)]$. This tells us that

$$\chi_I(x)\tilde{\chi}_0\Big(\big(h\frac{(x-y)}{mt}\big)^{m-1}\Big)\Big)\tilde{\Phi}(y)u_{j,0}(y) \neq 0$$

from $t = \frac{I_a-(U_b+2\varepsilon)}{(4h^{-1})^{m-1}}$ to $t = \frac{I_b-(U_a-2\varepsilon)}{(\frac{1}{4}h^{-1})^{m-1}}$. This implies that the integral with respect to $t$ is non-zero for at most $C/(h^{-1})^{m-1}$ time for a constant $C$.

$$
\begin{aligned}
A_j \leq & \frac{1}{2\pi}(1+4h^{-2})^{(m-1)/2} \int_0^{\frac{C}{(h^{-1})^{m-1}}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{1}{h} \chi_I(x) \tilde{\chi}_0 \left( h \left( \frac{(x-y)}{mt} \right)^{m-1} \right) \tilde{\Phi}(y) u_{j,0}(y) dy \right|^2 dxdt \\
& + \mathcal{O}(h^\infty)||u_{j,0}||_{L^2}^2 + \mathcal{O}(h^\infty)||u_0||_{L^2}^2 \\
\leq & \frac{1}{2\pi}(1+4(h^{-1})^2)^{(m-1)/2} \frac{C'}{(h^{-1})^{m-1}} ||u_{j,0}||_{L^2}^2 + \mathcal{O}((1/2^j)^\infty)||u_0||_{L^2}^2 \\
\leq & C''||u_{j,0}^+||_{L^2}^2 + \mathcal{O}((1/2^j)^\infty)||u_0||_{L^2}^2
\end{aligned}
$$

as $j \to \infty$ and $C'$ and $C''$ are positive constants that can be chosen independent of $j$. These estimates will hold for $\xi < 0$. We have

$$
\begin{aligned}
A_j^- &= \int_0^T \int_I |\langle D_x \rangle^{(m-1)/2} u_j^-(t,x)|^2 dxdt \\
&\leq C''||u_{j,0}^+||_{L^2}^2 + \mathcal{O}((1/2^j)^\infty)||u_0||_{L^2}^2.
\end{aligned}
$$

So,

$$
\begin{aligned}
\int_0^T \int_I |\langle D_x \rangle^{(m-1)/2} u(t,x)|^2 dxdt &\leq \sum_{j=0}^\infty A_j + \sum_{j=0}^\infty A_j^- + \int_0^T \int_I |\langle D_x \rangle^{(m-1)/2} \tilde{u}(t,x)|^2 dxdt \\
&\leq 2 \sum_{j=0}^\infty \left( C''||u_{j,0}||_{L^2}^2 + \mathcal{O}((1/2^j)^\infty)||u_0||_{L^2}^2 \right) + \int_0^T \int_I |\langle D_x \rangle^{(m-1)/2} \tilde{u}(t,x)|^2 dxdt.
\end{aligned}
$$

Now, $2C'' \sum_j ||u_{j,0}||_{L^2}^2 \leq K||u_0||_{L^2}^2$ and $\sum_{j=0}^\infty O((1/2^j)^\infty)||u_0||_{L^2}^2 \leq K'||u_0||_{L^2}^2$ for positive constants $K$ and $K'$. Additionally

$$
\int_0^T \int_I |\langle D_x \rangle^{(m-1)/2} \tilde{u}(t,x)|^2 dxdt \leq 2T||u_0||_{L^2}^2,
$$

since $|\langle \xi \rangle^{(m-1)/2} \hat{\tilde{u}}| \leq |\hat{u}|$. Combining the estimates gives

$$
\int_0^T \int_I |\langle D_x \rangle^{(m-1)/2} u(t,x)|^2 dxdt \leq C_T||u_0||_{L^2}^2
$$

for a positive constant $C_T$ as desired. □

## A.5 Information on the Potentials $V_{p,n}$

We will start by showing that the set $\{x|V'_{p,n}(x) = 0 \text{ for some } (p,n) \in \mathbb{Z} \times \mathbb{Z} \setminus (0,0)\}$ is dense in the interval $[-\varepsilon, \varepsilon]$ where $V_{p,n}(x)$ is defined in (4.3.1).

### A.5.1 Critical Points are Dense in $[-\varepsilon, \varepsilon]$

Let

$$x_{p,n} = \left(\frac{p^{2/(2m-1)} - n^{2/(2m-1)}}{p^{2/(2m-1)} + n^{2/(2m-1)}}\right).$$

Note that $V'_{p,n}(\varepsilon x_{p,n}) = 0$. The set $\{x_{p,n}|(p,n) \in \mathbb{Z} \times \mathbb{Z} \setminus (0,0)\}$ is dense in $[-1,1]$. Let $x \in [-1,1]$, $\delta > 0$, and $k = 2/(2m-1)$.

If $x = -1$, then $|x - x_{n,0}| = 0 < \delta$ for $p \neq 0$. If $x = 1$, then $|x - x_{p,0}| = 0 < \delta$ for $n \neq 0$.

If $x \in (-1,1)$, let $p = An$. Then,

$$|x - x_{p,n}| = |((A^k+1)x - (A^k-1))/(A^k+1)| = |(A^k(x-1) + (x+1))/(A^k+1)|.$$

Then $A^k = \frac{x+1}{1-x} > 0$ gives $|x - x_{p,n}| = 0$. However, $\frac{x+1}{1-x}$ could be irrational.

Let $|B - \frac{x+1}{1-x}| < \frac{\delta}{2}$. Then,

$$|(B(x-1) + (x+1))/(A^k+1)| < \left|\left(\frac{x+1}{1-x} + \frac{\delta}{2}\right)(x-1) + (x+1)\right| \leq \left|\frac{\delta}{2}(x-1)\right| < \delta.$$

Now, $f(x) = x^k$ is a continuous function on $(0,\infty)$ and rational numbers are dense in $\mathbb{R}$. This implies for a given $\delta > 0$ that for all $\delta' > 0$ there exists $q, r \in \mathbb{Z}$ such that $|(\frac{x+1}{1-x})^{1/k} - q/r| < \delta'$ and for $\delta' > 0$ sufficiently small, we have $|(\frac{x+1}{1-x}) - (q/r)^k| < \delta/2$. So, by choose $q, r$ so that $\delta'$ is sufficiently small we get

$$|x - x_{q,r}| = |((A^k+1)x - (A^k-1))/(A^k+1)| < \left|\left(\frac{x+1}{1-x} - \delta/2\right)(x-1) + (x+1)\right| < \delta.$$

Multiplying $x_{p,n}$ by $\varepsilon$ completes the proof that

$$\{x|V'_{p,n}(x) = 0 \text{ for some } (p,n) \in \mathbb{Z} \times \mathbb{Z} \setminus (0,0)\}$$

is dense in $[-\varepsilon, \varepsilon]$.

### A.5.2 Nature of the critical points

Recall from Figure 2.3 we discussed the level sets of the Hamiltonian. In this case our Hamiltonian will be given by $\xi^2 + h^2 V_{p,n}(x)$. We can look at level sets if we fix the values of $p$ and $n$. Figure A.3 is an sketch of when the Hamiltonian is 0 for $p = 0$ and $n = 0$. Notice that this is the situation where we have degenerate unstable critical points. However, we can have the situation where $p, n \neq 0$.

In Figure A.4 we have the situation when $p = n$ and when $n \gg p$. In this situation both of the critical points will be non-degenerate unstable critical points. However, when $n \gg p$ we approach the degenerate unstable critical point given in Figure A.3. This combination of the behavior when $n/p \to \infty$ and when $p = n$ makes the resolvent estimate difficult. We expound on this in the next subsection.

Lets consider the case where $m = 2$ and we will use the calculations from Subsection A.5.3. In this situation the level sets given in Figure A.3 and A.4 are roughly given by $\xi^2 = h^{(2-2\eta)2/3}(x - x_{p,n})^2 + (x - x_{p,n})^4$ where $h^{-2} = p^2 + n^2$ and $p \sim h^{-\eta}$ where $n > p$. Near the critical point the level sets are approximately given by $\pm\xi \approx h^{(2-2\eta)/3}(x - x_{p,n})$ if $h \neq 0$. What makes the microlocal resolvent estimate in (4.4.12) hard is that while the critical point is non-degenerate, as long as $\eta > 0$, the critical point acts like a degenerate point as $h \to 0$.

### A.5.3 Estimates of the Derivatives of the potential

In this section we show some estimates on the higher order derivatives of the potential. These estimates will not be used to get the local smoothing result, however they provide

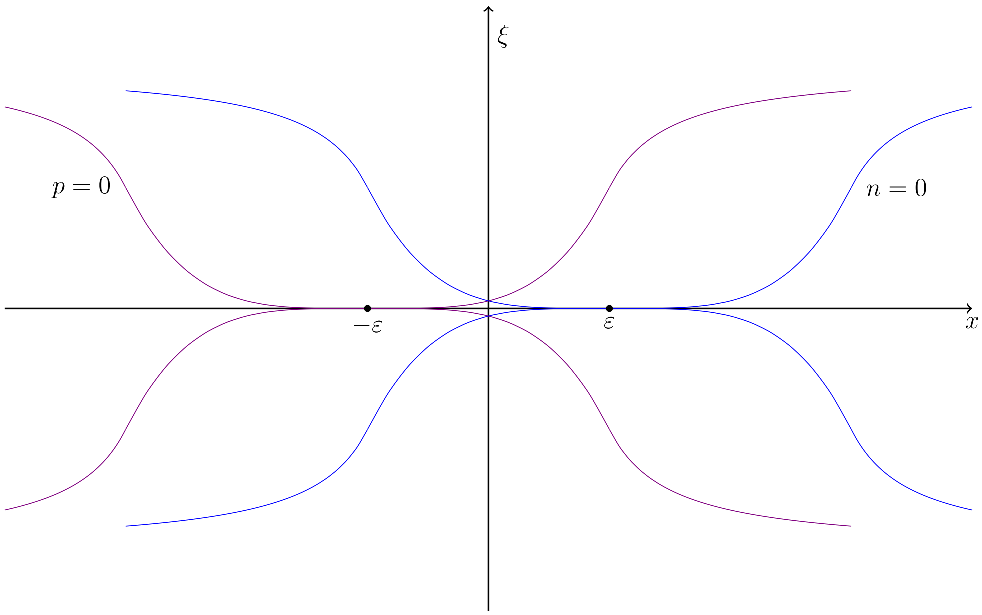

Figure A.3: Level sets of $\xi^2 + h^2 V_{p,n}(x)$ for $p = 0$ and $n = 0$

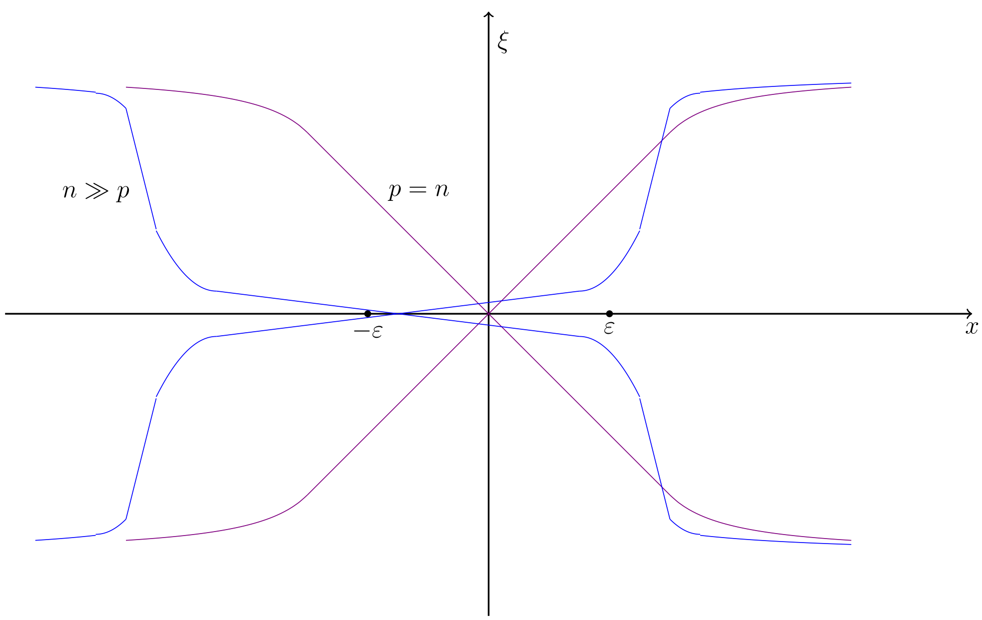

Figure A.4: Level sets of $\xi^2 + h^2 V_{p,n}(x)$ for $p = n$ and $n \gg p$

a heuristic for why this estimate is tough. They also show why we will need to employ a two-parameter calculus to get the final estimate.

Recall $x_{p,n} = \varepsilon \left( \frac{p^{2/(2m-1)} - n^{2/(2m-1)}}{p^{2/(2m-1)} + n^{2/(2m-1)}} \right)$ is the critical point of of $V'(x)$. Suppose $h$ is sufficiently small with $h^2 = \frac{1}{p^2+n^2}$ and $|n| \leq |p|$ and $k > 1$ is an integer. If $|n| \geq h^{-\eta}$ with $1 - \delta > \eta > 0$

for $\delta > 0$ small, then there exists a $C' > 0$ independent of $h$ (once $h$ is sufficiently small) such that

$$|h^2 V_{p,n}^{(k)}(x_{p,n})| \geq C' h^{(2-2\eta)(2m-k)/(2m-1)}. \tag{A.5.1}$$

If $|n| \leq h^{-\eta}$ then there exists a $C > 0$ independent of $h$ such that

$$|h^2 V_{p,n}^{(k)}(x_{p,n})| \leq C h^{(2-2\eta)(2m-k)/(2m-1)}. \tag{A.5.2}$$

We will show these estimates and then explain why (A.5.1) is an issue to proving the local smoothing estimate.

$$V_{p,n}^{(k)}(x) = -\left(\frac{2m!}{(2m-k)!}\right)p^2(x-\varepsilon)^{2m-k} - \left(\frac{2m!}{(2m-k)!}\right)n^2(x+\varepsilon)^{2m-k}.$$

At the critical point

$$\begin{aligned} &V_{p,n}^{(k)}(x_{p,n}) \\ &= \left(-\frac{2m!}{(2m-k)!}\varepsilon^{2m-k}\right)\left(p^2\left(\frac{-2n^{2/2m-1}}{p^{2/2m-1}+n^{2/2m-1}}\right)^{2m-k} + n^2\left(\frac{2p^{2/2m-1}}{p^{2/2m-1}+n^{2/2m-1}}\right)^{2m-k}\right). \end{aligned}$$

Replacing the constants dependent on $\varepsilon$ and $k$ and $m$ with $C$ we get

$$V_{p,n}^{(k)}(x_{p,n}) = C\left(\frac{n^2 p^{(4m-2k)/(2m-1)} + (-1)^{2m-k} p^2 n^{(4m-2k)/(2m-1)}}{(p^{2/(2m-1)} + n^{2/(2m-1)})^{2m-k}}\right)$$

First, let $|n| \geq h^{-\eta}$. Note that $|p| \sim 1/h$, but the exact relationship is dependent on $n$. However, $1/h \geq |p| \geq 1/(2h)$ for all $|n| \leq |p|$. Then,

$$\begin{aligned} |V_{p,n}^{(k)}(x_{p,n})| &\geq \left|\frac{C_1 h^{-2\eta} h^{-(4m-2k)/(2m-1)} + (-1)^{2m-k} C_2 h^{-2} h^{-\eta(4m-2k)/(2m-1)}}{(2h^{-2/(2m-1)})^{2m-k}}\right| \\ &\geq C'(h^{-2} h^{-\eta(4m-2k)/(2m-1)})(h^{(4m-2k)/(2m-1)}) \\ &\geq C' h^{-2} h^{(2-2\eta)(2m-k)/(2m-1)} \end{aligned}$$

for $h$ sufficiently small and constants $C_1, C_2$ and $C'$ since the $h^{-2}h^{-\eta(4m-2k)/(2m-1)}$ term dominates as $h \to 0$ because $k > 1$. This gives the estimate in (A.5.1).

Now, let $|n| \leq h^{-\eta}$. Then,

$$\begin{aligned} V_{p,n}^{(k)}(x_{p,n}) &\leq C \frac{h^{-2\eta}h^{-(4m-2k)/(2m-1)} + h^{-2}h^{-\eta(4m-2k)/(2m-1)}}{(h^{-2/(2m-1)})^{2m-k}}, \\ &\leq C' h^{-2}h^{-\eta(4m-2k)/(2m-1)}h^{(4m-2k)/(2m-1)} \\ &\leq C' h^{-2}h^{(2-2\eta)(2m-k)/(2m-1)}. \end{aligned}$$

for positive constants $C$ and $C'$.

Recall in Section 3.3 and (3.3.14) that we discussed a commutator argument with $[a^w, Q_1]$ where $Q_1 = (hD_x)^2 + V - h^2V_1$ for $V$ and $V_1$. Then $[a^w, Q_1] = h\{a, q_1\}^w + h^3r^w$ where $r$ is dependent on the odd order derivatives of $a$ and $q_1$, where $q_1$ is the symbol for $Q_1$. In the multi-warped product case here, we will have the $Q = (hD_x)^2 + h^2V_{p,n} + \mathcal{O}(h^2)$ where $-V_{p,n} = p^2(x-\varepsilon)^{2m} + n^2(x+\varepsilon)^{2m}$. If we consider the case where $m = 2$, then $h^3r \sim h^3(\partial_\eta^3 a h^2 V^{(3)}) \sim h^{2/3}(h^2V^{(3)})$. From [CW13], we expect and will show that the optimal lower bound is $\langle h\{a, q_1\}^w u, u\rangle \geq h^{4/3}||u||_{L^2}^2$. However, from (A.5.1)

$$|h^{2/3}(h^2V^{(3)})| \geq h^{2/3}h^{(1-\eta)2/3} > h^{4/3}$$

if $1 > \eta > 0$. This makes estimating the higher order derivatives difficult and is the main challenge in proving Theorem 4.2.2 for $m \geq 2$. The case of $m = 2$ was possible because we were able show

$$h^2V_{p,n}'(x)(x - x_{p,n}) \geq C\left(\frac{E_1}{E_3}(x - x_{p,n})^2 + (x - x_{p,n})^4\right)$$

in Lemma 4.4.26. This allowed us to bound the error term by $h^2V_{p,n}'(x)(x - x_{p,n})$ rather than just $h^{4/3}$. The key to this estimate was handling the cubic term of $V_{p,n}$ when doing a Taylor expansion, since it can have the wrong sign.

# BIBLIOGRAPHY


[Bur04] Nicolas Burq, *Smoothing effects of Schrödinger boundary value problems*, Duke Mathematics Journal **123** (2004), no. 2, 403-427. ↑5, 36

[Chr07] Hans Christianson, *Semiclassical Non-concentration near Hyperbolic Orbits*, Communications in Partial Differential Equations **246** (2007), no. 3, 145-195. ↑5

[Chr08] ______, *Dispersive Estimates for Manifolds with one Trapped Orbit*, Communications in Partial Differential Equations **33** (2008), no. 7, 1147–1174. ↑5, 36, 66

[Chr18] ______, *High-frequency resolvent estimates on asymptotically Euclidean warped products*, Communications in Partial Differential Equations **43** (2018), no. 9, 1306-1362. ↑36, 43, 83

[CM14] Hans Christianson and Jason Metcalfe, *Sharp Local Smoothing for Warped Product Manifolds with Smooth Inflection Transmission*, Indiana University Mathematics Journal **63** (2014), no. 4, 969-992. ↑iii, vii, 6, 7, 12, 36, 50, 51, 53, 54, 81, 90

[CN22] Hans Christianson and Derrick Nowak, *Local Smoothing for the Schrödinger Equation on a Multi-Warped Product Manifold with Inflection-Transmission Trapping*, Proceedings of the American Mathematical Society **150** (2022), no. 1, 213-230. ↑iii, 6, 7, 10, 18, 36, 52, 53, 54, 55, 58

[CW13] Hans Christianson and Jared Wunsch, *Local smoothing for the schrodinger equation with prescribed loss*, American Journal of Mathematics **135** (2013), no. 6, 1601-1632. ↑iii, 6, 7, 12, 25, 26, 36, 42, 43, 48, 49, 50, 51, 53, 54, 66, 75, 81, 83, 86, 90, 94, 117

[CKS95] Walter Craig, Thomas Kappeler, and Walter Strauss, *Microlocal dispersive smoothing for the Schrödinger equation*, Comm. Pure Appl. Math **48** (1995), no. 8, 769-860. ↑5, 36

[CS88] Peter Constantin and Jean-Claude Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc. **1** (1988), no. 2, 413-439. ↑5

[Dat09] Kiril Datchev, *Local smoothing for scattering manifolds with hyperbolic trapped sets*, Communications in Mathematical Physics **286** (2009), no. 3, 837-850. ↑5, 36

[Doi96] Shin-ichi Doi, *Smoothing effects of Schrödinger evolution groups on Riemannian Manifolds*, Duke Mathematics Journal **82** (1996), no. 3, 679-706. ↑5, 36

[Hör07] Lars Hörmander, *The Analysis of Linear Partial Differential Operators III*, Classics in Mathematics, Springer Berline, Heidelberg, 2007. ↑20, 21

[LPG+19] Anna Lischke, Guofei Pang, Mamikon Gulian, Songm Fangying, Christian Glusa, Xiaoning Zheng, Zhiping Mao, Wei Cai, Mark M. Meerschaert, Mark Ainsworth, and George Em Karniadakis, *What Is the Fractional Laplacian?*, 2019. ↑27

[RS81] Michael Reed and Barry Simon, *I: Functional Analysis*, Elsevier Science, 1981. ↑27, 28

[JSS91] J.-L. Journe, Avy Soffer, and Christopher Sogge, *Decay estimates for Schrödinger operators*, Communications on Pure and Applied Mathematics **44** (1991), no. 4, 573-604. ↑5

[Kat83] Tosio Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies in Applied Mathematics: a volume dedicated to Irving Segal **8** (1983), 93–128. ↑5

[KY89] Tosio Kato and Kenji Yajima, *Some examples of smooth operators and the associated smoothing effect*, Rev. Math. Phys. **1** (1989), no. 4, 481–496. ↑5

[MMT08] Jeremy Marzuola, Jason Metcalfe, and Daniel Tataru, *Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations*, J. Funct. Anal. **255** (2008), no. 6, 1497-1553. ↑5, 36

[Ral71] James Ralston, *Trapped rays in spherically symmetric media and poles of the scattering matrix*, Communications on Pure and Applied Mathematics **24** (1971), 571-582. ↑5, 36

[RT07] Igor Rodnianski and Terence Tao, *Longtime decay estimates for the Schrödinger equation on manifolds*, Annals of Mathematics Studies **163** (2007), 223-253. ↑5, 36

[Sjö87] Peter Sjölin, *Regularity of solutions to the Schrodinger equation*, Duke Math. J. **55** (1987), no. 3, 699-715. ↑5

[SZ07] Johannes Sjöstrand and Maciej Zworski, *Fractal upper bounds on the density of semi-classical resonances*, Duke Math Journal **137** (2007), no. 3, 381-459. ↑25

[Tao06] Terence Tao, *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006. Local and global analysis. ↑

[Veg88] Luis Vega, *Schrödinger equations: pointwise convergence to the initial data*, Proc. Amer. Math. Soc. **102** (1988), no. 4, 874–878. ↑5

[Zwo12] M. Zworski, *Semiclassical Analysis*, Graduate studies in mathematics, American Mathematical Society, 2012. ↑4, 22, 23, 24, 103